Oldies: (which does not necessarily mean goldies :-) Pomset logic, proof-nets and coherence semantics

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Warning and apologies

- Old work, nineties
  (with little time to re-work it)
- Special thanks to Sylvain Pogodalla,
  who spent part of his thesis to
  try to prove with me yet unsolved questions
- Motivated by a possibility to solve open questions:
  - More fashionable
    sequent/term/rewrite caculus
  - Correspondence with BV

Coherence Semantics

- Formulae: (possibly infinite) graphs
- Proofs up to normalisation: cliques
- Morphisms, linear maps:
  - F sends cliques to cliques
  - When a union is a clique:
    - Commute with union
    - Commute with intersection

Multiplicative coherence spaces Girard's remark

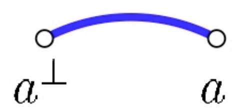
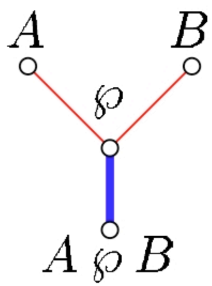
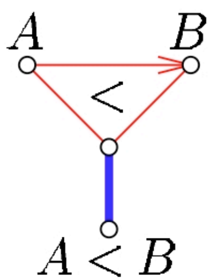
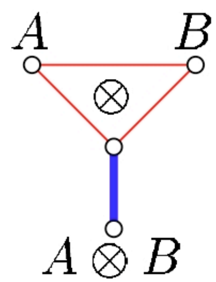
- Vertices: pairs of vertices
- Par: both $\frown$
- Times: both $\smile$
- One non commutative« < »: A: $\smile$ and B: $\frown$
- No other multiplicative.

$A \backslash B$	$\frown$	$=$	$\smile$
$\frown$	$\frown$	$\frown$	?
$=$	$\frown$	$=$	$\smile$
$\smile$	?	$\smile$	$\smile$

Before

- Written <
  - Non commutative
  - Associative
  - Self-dual $(A < B)^{\perp} \equiv (A^{\perp} < B^{\perp})$
- Girard's question:
  what syntax for this calculus?

Bicoloured proof nets

Name	*axiom*-link	*par*-link	*before*-link	*times*-link
Premises	none	A and B	A and B	A and B
R&B-graph				
Conclusions	a and $a^\perp$	$A \wp B$	$A < B$	$A \otimes B$

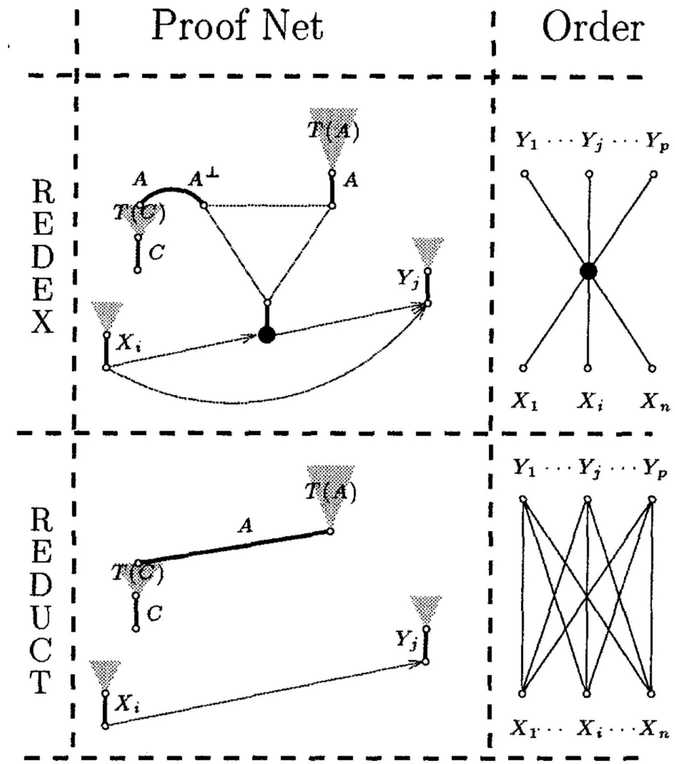
INR

Proof nets

- Extra-arc for denoting an order (preferably SP, definable) between conclusions
- Criterion no alernate elementary cycle
- Viewing cuts as $(\exists K) K \otimes K$ they take part in the order

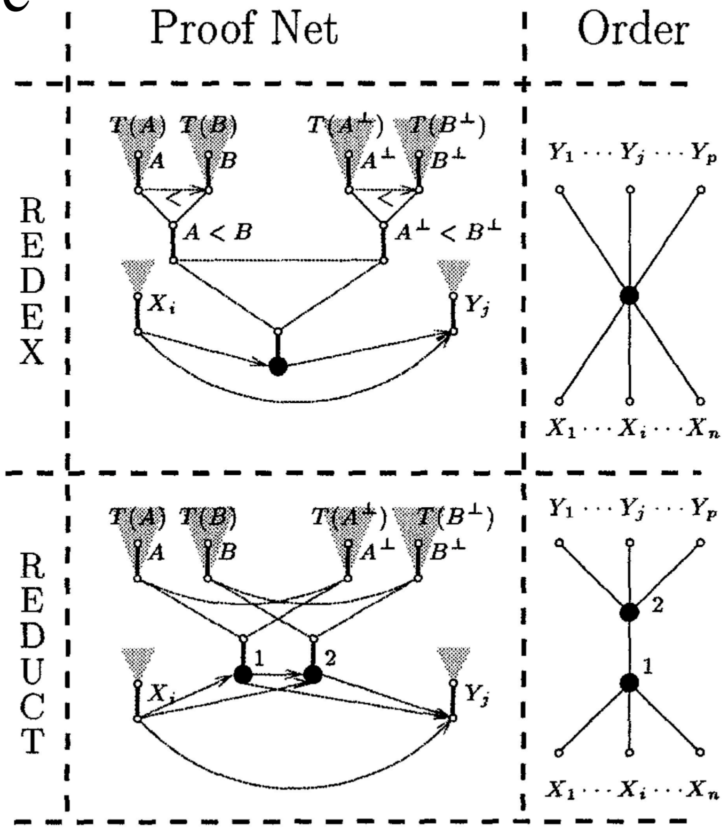
Cut elimination perserves correctness

Cut on axiom



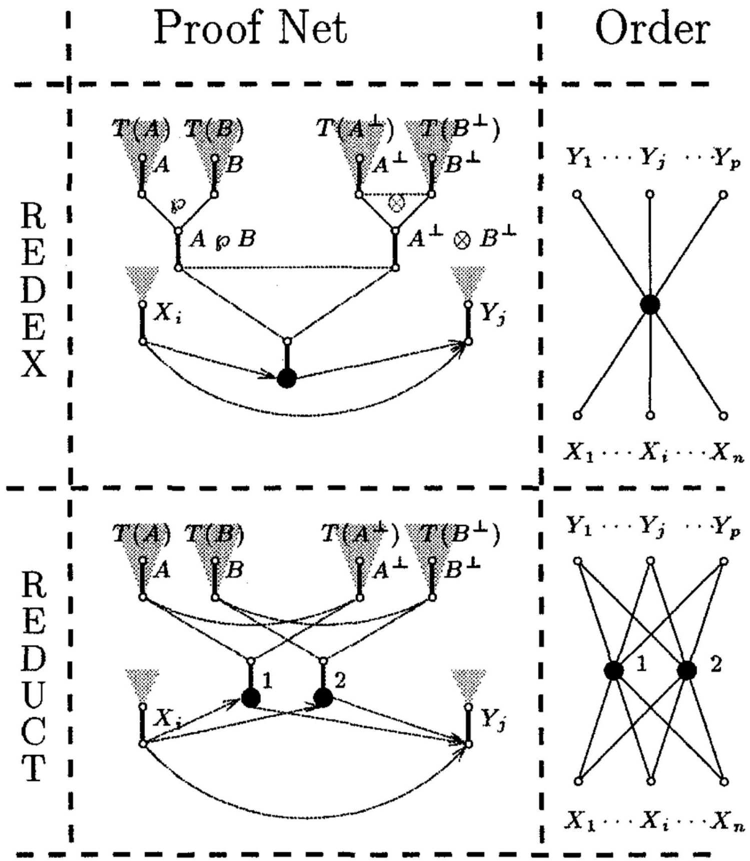
Cut elimination perserves correctness and order

Cut before/before



Cut elimination perserves correctness and order

Cut times/par



Interpreting proofs

- Choose a token for each axiom
- Collect the tuples: they are a clique of the coherence space associated with the partially ordered set of conclusions:

$$\vec{x} \smile \vec{y}[(A_i)_{i \in (I,<)}]$$
$$\Leftrightarrow$$
$$\exists i\, x_i \smile y_i \wedge (\forall j > i\; x_j = y_j)$$

Interpreting proofs: soundness and « completeness »

- Proof: would lead to an infinite alternate elementary path incoherent moving up, coherent moving down.
- Moreover the converse is true: if the proofnet is not correct, some interpretations are not cliques even in a single finite coherence space: N (isomorphic to its orthogonal Z)

Directed cographs

- Directed cographs for denoting formulae:
  - Containing the single vertex graphs
  - Closed under
    - Disjoint union
    - Undirected series composition
    - Directed series composition
    - (Hence under complementation if an undirected edge is viewed a pair of opposite directed edges)

Directed cographs

- Universal characterisation:
  - The directed part is an SP order
  - The undirected part is a cograph
  - Weak transitivity

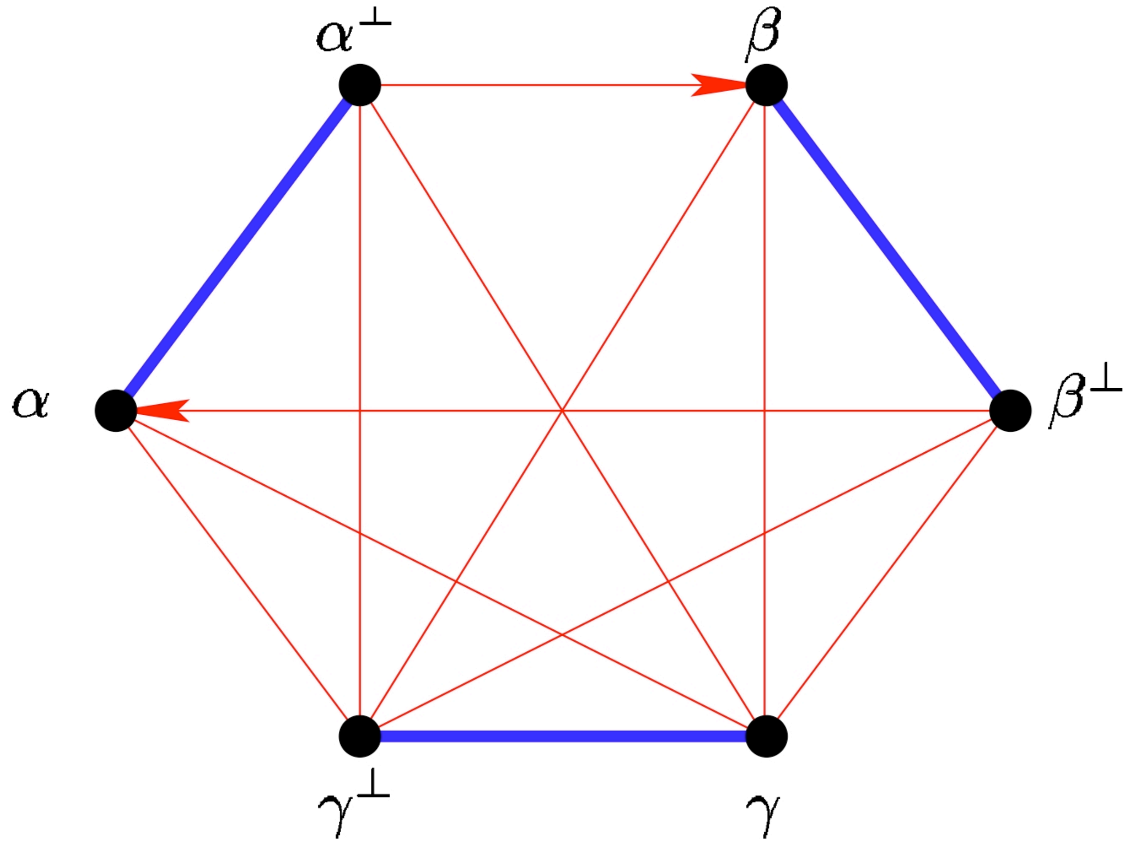
$$(x,y) \in R \wedge (y,x) \notin R \wedge (y,z) \in R \Rightarrow (x,z) \in R$$

$$(x,y) \in R \wedge (y,z) \in R \wedge (z,y) \notin R \Rightarrow (x,z) \in R$$

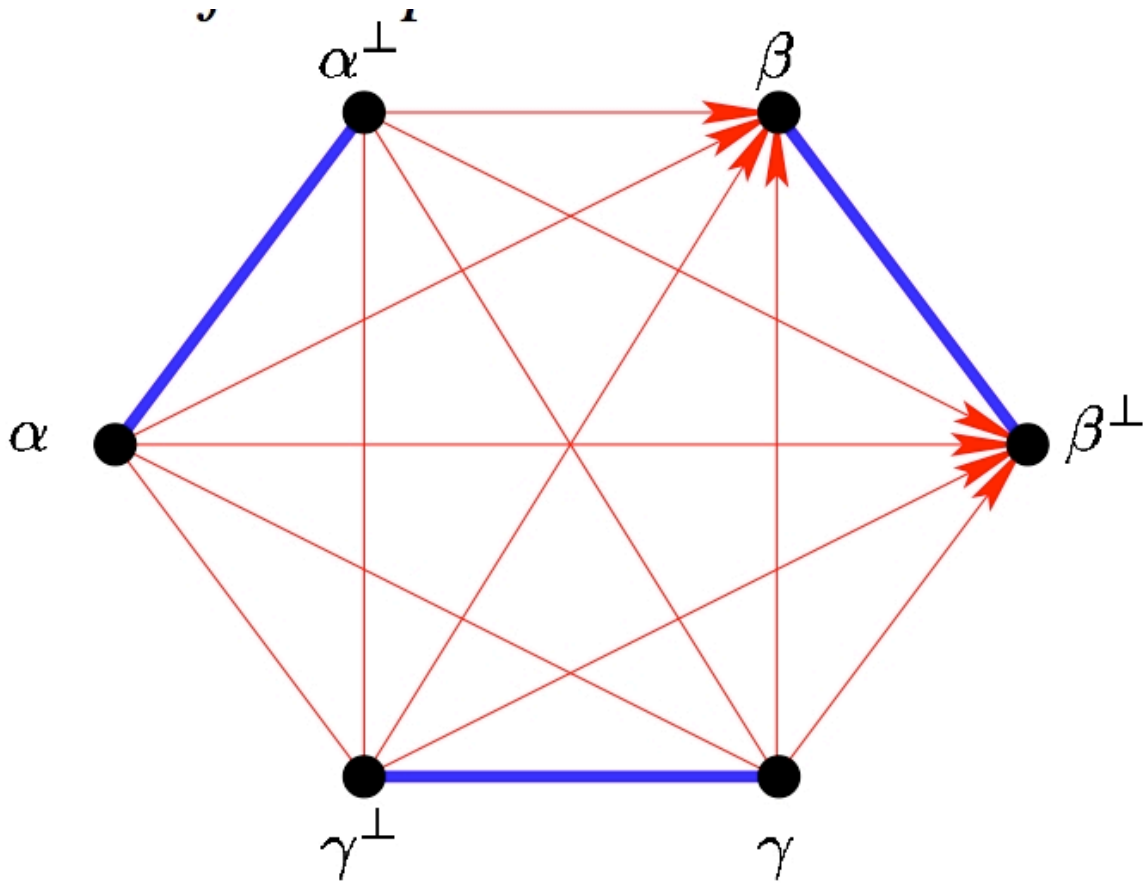
Handsome proofnets

- Vertices: propositional variables and their negations
- A directed cograph (the formula)
- Plus a perfect matching (the axioms)
- Criterion:
  – Every alternate elementary cycle contains a chord

Uncorrect

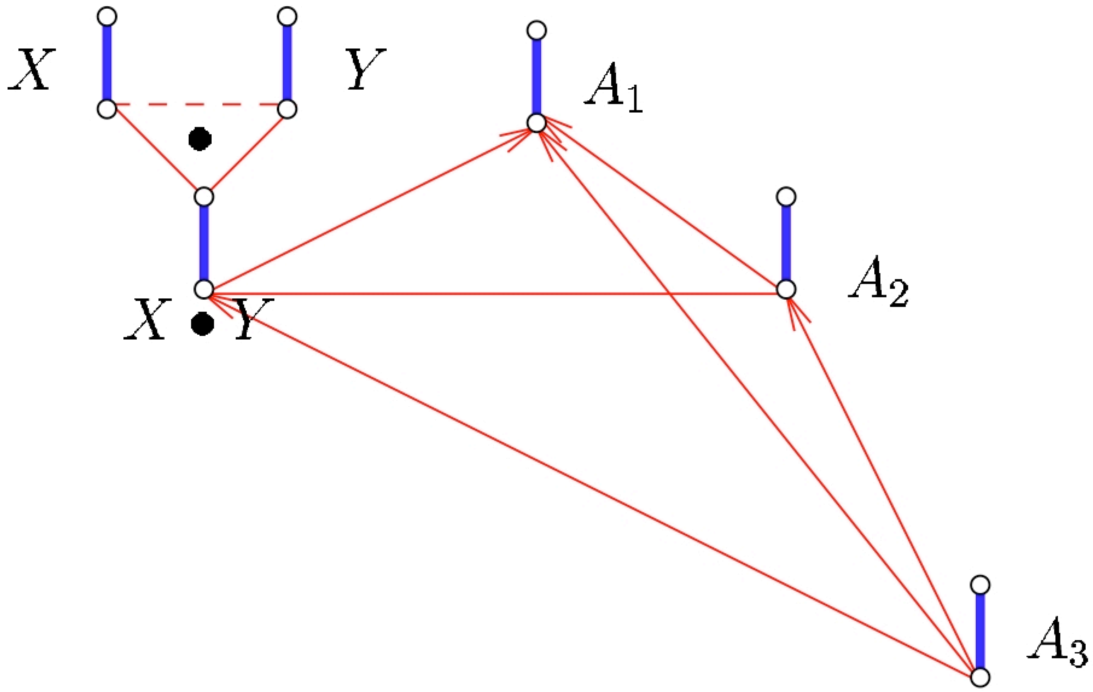


Correct



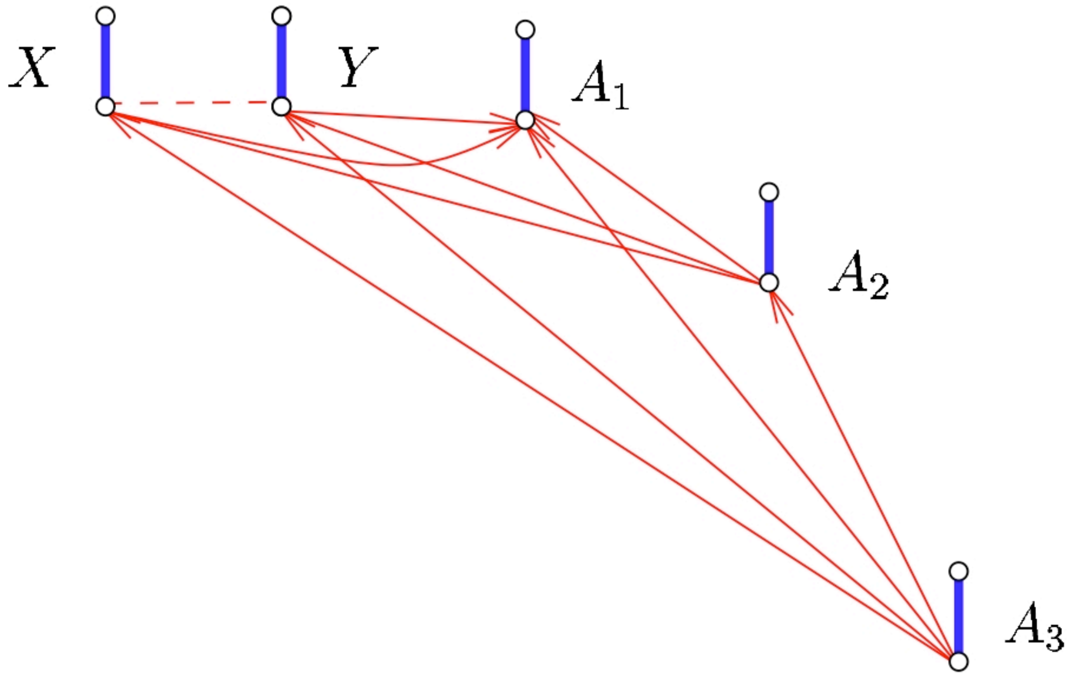
Fold

$\boxed{\Pi_\bullet}$

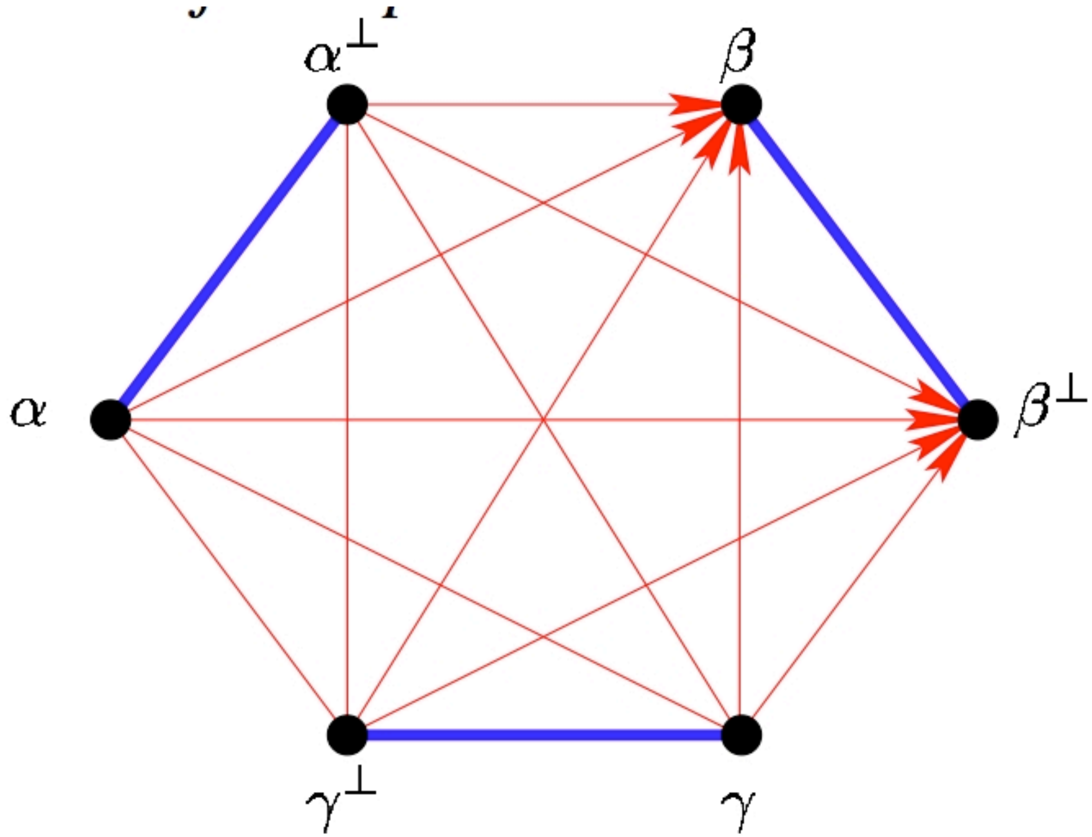


Unfold

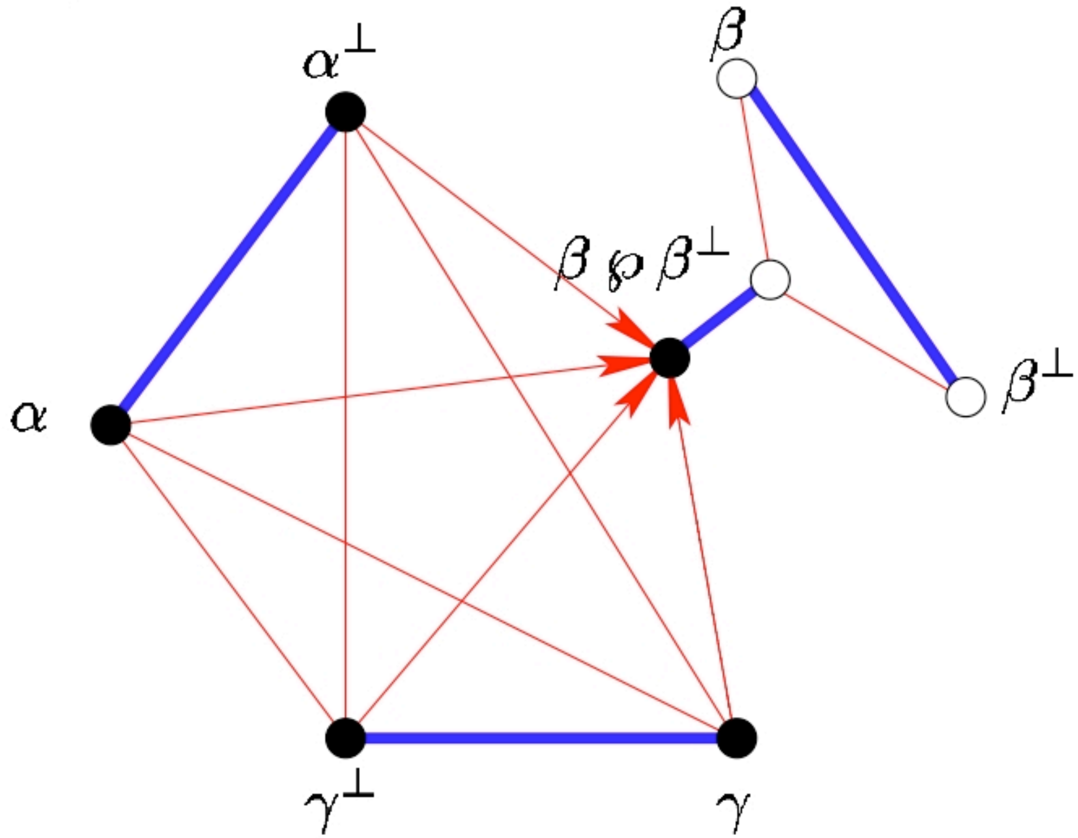
$\boxed{\Pi_{\hat{\bullet}}}$



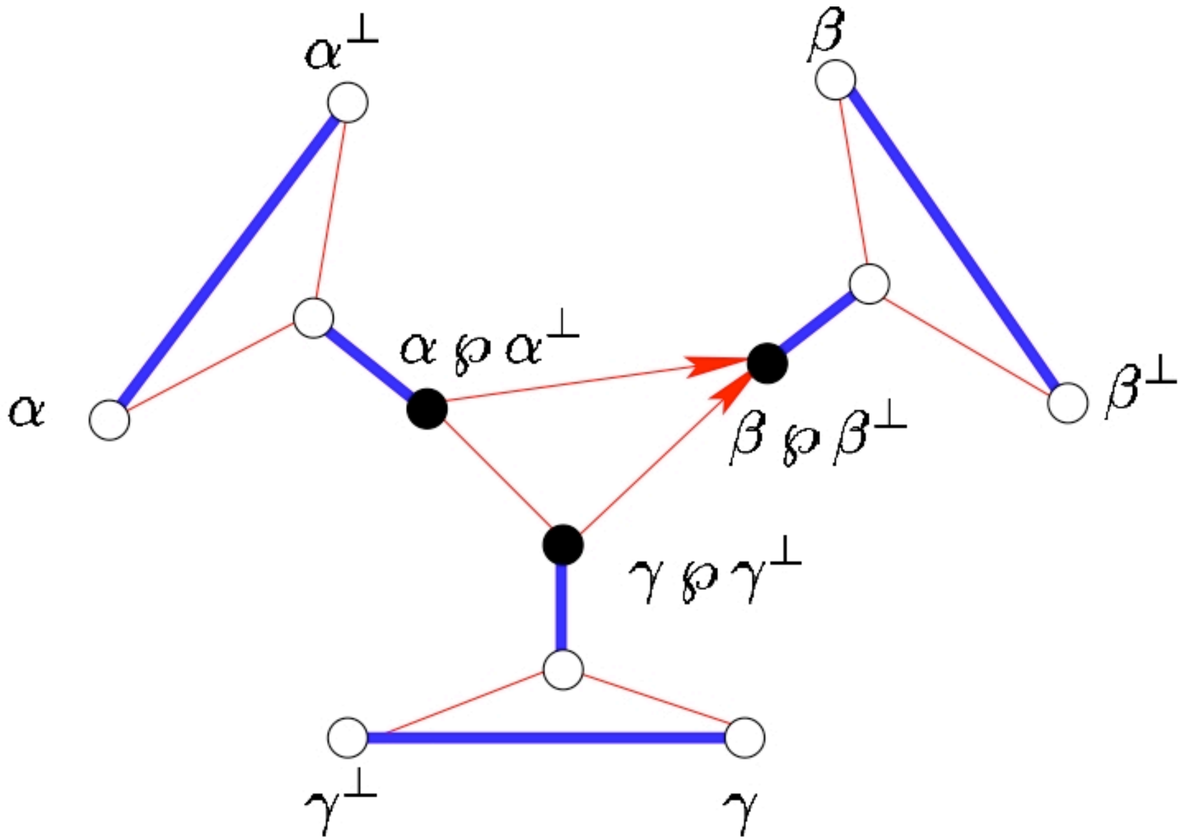
Correct



Correct with a link



Correct with three links



Property

- Fold and unfold preserve the criterion that every alternate lementary cycle contains a chord.
- Observe that when there are only links, this means that there is no alternate elementary cycle at all.

Cut-elimination

- Works directly on axioms
- Also derives from the one on proof nets with links.
- Looks like Girard’s turbo cut-elimination

Rewriting (black lollipop preserves correctness)

$(\otimes\wp4)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{\otimes}$	$(U \mathbin{\widehat{\wp}} V)$	$\longrightarrow$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{\wp}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes\wp3)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{\otimes}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{\wp}$	Y
$(\otimes\wp2)$	Y	$\widehat{\otimes}$	U	$-\!\!\bullet$	U	$\widehat{\wp}$	Y
$(\otimes<4)$	$(X \mathbin{\widehat{<}} Y)$	$\widehat{\otimes}$	$(U \mathbin{\widehat{<}} V)$	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{<}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes<l3)$	$(X \mathbin{\widehat{<}} Y)$	$\widehat{\otimes}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{\otimes}} U)$	$\widehat{<}$	Y
$(\otimes<r3)$	Y	$\widehat{\otimes}$	$(U \mathbin{\widehat{<}} V)$	$-\!\!\bullet$	U	$\widehat{<}$	$(Y \mathbin{\widehat{\otimes}} V)$
$(\otimes<2)$	Y	$\widehat{\otimes}$	U	$-\!\!\bullet$	U	$\widehat{<}$	Y
$(<\wp4)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{<}$	$(U \mathbin{\widehat{\wp}} V)$	$-\!\!\bullet$	$(X \mathbin{\widehat{<}} U)$	$\widehat{\wp}$	$(Y \mathbin{\widehat{<}} V)$
$(<\wp l3)$	$(X \mathbin{\widehat{\wp}} Y)$	$\widehat{<}$	U	$-\!\!\bullet$	$(X \mathbin{\widehat{<}} U)$	$\widehat{\wp}$	Y
$(<\wp r3)$	Y	$\widehat{<}$	$(U \mathbin{\widehat{\wp}} V)$	$-\!\!\bullet$	U	$\widehat{\wp}$	$(Y \mathbin{\widehat{<}} V)$
$(<\wp2)$	Y	$\widehat{<}$	U	$-\!\!\bullet$	U	$\widehat{\wp}$	Y

Conjecture

- All correct handsome proofnets are obtained by the correct rewriting from

$$\bigotimes_i (a_i \wp a_i^{\perp})$$

- (True for MLL)

Sequent calculus?

- Times as usual
- Par as usual
- MIX introduces the order
  the restrictions of K to G and D should
  be I and J

$$\frac{\vdash \Gamma[I] \qquad \vdash \Delta[J]}{\vdash \Gamma, \Delta[K]}$$

- Yields all correct proof nets?

Alternative conjecture (would directly yield sequentialisation)

- Given a correct handsome proofnet, there exists a partition A_1 A_2 of the axiom links (hence a partition V_1 V_2 of the vertices, since they are a complete matching) such that:
  - All the crossing edges are undirected and define a complete bipartite graph $K(U_1,U_2)$ with U_1 included in V_1 and U_2 included in V_2
  - All the crossing edges are directed and they all go from V_1 to V_2 or they all go from V_2 to V_1.

Old references

- 1993 Réseaux et séquents ordonnés PhD Thesis Paris 7
- 1997 Pomset logic a non commutative extension of classical linear logic. TLCA
- 1997 (with Bechet and de Groote) A complete axiomatisation of the inclusion a SP orders. RTA
- 1997 A semantic characterisation of the correctness of a proof nets. MSCS / INRIA Report
- 2003 Handsome proofnets: perfect matchings and cographs. TCS / INRIA Report