(Nearly-)Tight Bounds on the Linearity and Contiguity of Cographs

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Extended Abstract

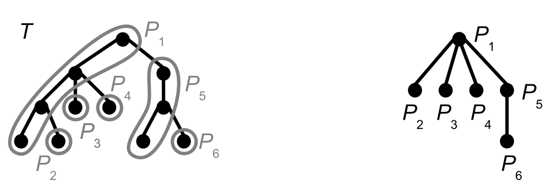
Introduction. Linearity and contiguity are graph parameters introduced to obtain efficient codings of neighborhoods in graphs, by decomposing each neighborhood as a union of p intervals chosen in one or several orders on the vertices [1]. Indeed, storing an order of the vertices as well as a pair of pointers for each of the p intervals of this order (one pointer for the beginning of the interval and one for the end), with fixed p, allows to store the graph in $O(n)$ space (instead of $O(n+m)$ with adjacency lists) and access the neighborhood of any vertex v in $O(d)$ time (instead of $O(n)$ with adjacency matrices), where d is the degree of v.

More formally, a *closed p-interval-model* of a graph $G = (V, E)$ is a linear order σ on V such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in \int 1, p,$ I_i is an interval of σ and $N[x] = \bigcup_{1 \leq i \leq p} I_i$. The *closed contiguity* of G, denoted by $cont(G)$, is the minimum integer p such that there exists a closed p-interval-model of G. A *closed p-line-model* of a graph $G = (V, E)$ is a tuple $(\sigma_1, \ldots, \sigma_p)$ of linear orders on V such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in \int 1, p,$ I_i is an interval of σ_i and $N[x] = \bigcup_{1 \leq i \leq p} I_i$. The *closed linearity* of G, denoted by $lin(G)$, is the minimum p such that there exists a closed p-line-model of G.

Not much is known about these parameters, which cannot be bounded by a constant even in very restricted graph classes, like interval or permutation graphs [1]. We focus here on the contiguity and linearity of cographs (graphs without induced P_4 subgraphs), whose very constrained structure can be represented by their *cotree*, a rooted tree with two kinds of nodes labeled by P and S, giving a tight upper bound for the asymptotic contiguity of cographs and an upper bound for their linearity. To this aim, we first establish a min-max theorem on the link between the rank of rooted trees and their decompositions into paths.

A min-max theorem on the rank of a tree. The *rank* [2, 3] of a tree T is the maximal height of a complete binary tree obtained from T by edge contractions, that is $rank(T) = \max\{h(T') \mid T' \text{ complete binary tree, minor of } T\}$.

A *path partition* of a tree T is a partition $\{P_1, \ldots, P_k\}$ of $V(T)$ such that for any i, the subgraph $T[P_i]$ of T induced by P_i is a path, as shown in Figure 1(a). The *partition tree* of a path partition $\mathcal{P}$, denoted by $T_p(\mathcal{P})$ and illustrated in Figure 1(b), is the tree whose nodes are P_i's and where the node of $T_p(\mathcal{P})$ corresponding to P_i is the parent of the node corresponding to P_j iff some node of P_i is the parent in T of the root of P_j. The height of a path partition $\mathcal{P}$ of a tree T, denoted by $h(\mathcal{P})$, is the height $h(T_p(\mathcal{P}))$ of its partition tree. The *path-height* of T is the minimal height of a path partition of T, that is $ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P} \text{ path partition of } T\}$.



(a) (b)

Figure 1: A tree T and a path partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ of T (a), as well as the partition tree of $\mathcal{P}$ (b).

Lemma 1 *For a rooted complete binary tree T, $rank(T) = ph(T) = h(T)$.*

Theorem 2 *For any rooted tree T, we have $rank(T) = ph(T)$.*

Upper bounds for contiguity and linearity of cographs. We now combine the results of the previous section with a decomposition of the cotree of the input cograph into paths, in order to obtain a constructive proof that the contiguity of any cograph is at most $O(\log n)$. This decomposition is obtained recursively, using a root-path decomposition of the cotree, thanks to the Caterpillar Composition Lemma below.

A *root-path decomposition* (see Fig. 2) of a rooted tree T is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of T, with $p \geq 2$, such that every leaf of T belongs to some T_i, with $i \in [1..p]$, and the sets of parents in T of the roots of T_i's is a path containing the root of T.

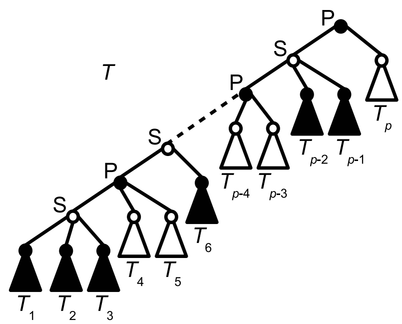


Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree T.

Lemma 3 (Caterpillar Composition Lemma) *Given a cograph $G = (V, E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where X_i is the set of leaves of T_i, $cont(G) \leq 2 + \max\limits_{i \in [1..p]} cont(G[X_i])$.*

Lemma 4 *Given a rooted tree T such that $rank(T) = k \geq 1$, there exists a root-path decomposition $\{T_1, \ldots, T_p\}$ of T such that for each $i \in [1..p]$, $rank(T_i) \leq k - 1$.*

Lemma 5 *Let G be a cograph and T its cotree. We have $cont(G) \leq 2\,rank(T) + 1$.*

Theorem 6 *The closed contiguity of a cograph is at most logarithmic in its number of vertices, or more formally, if $G = (V, E)$ is a cograph, then $cont(G) \leq 2 \log_2 |V| + 1$.*

Lower bounds for contiguity and linearity of cographs. Finally, we focus on cographs whose cotrees are complete binary trees, and obtain a tight lower bound for their asymptotic contiguity as well as a lower bound for their asymptotic linearity.

Theorem 7 *Let G be a cograph whose cotree is a complete binary tree. Then, $cont(G) = \Omega(\log n)$ and $lin(G) = \Omega(\log n / \log \log n)$.*

References

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