A $\frac{3}{2}$ dual approximation for the problem of minimazing makespan when scheduling independent tasks on heterogeneous processors

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Definition of the problem

Using the standard notation, this problem is $Q||C_{max}$

- Offline monocriteria optimization problem
- Input :
  - m machines of speed $s_1 \leq s_2 \leq ... \leq s_m$, $s_i > 0$
  - n jobs with processing requirement $p_1 \leq ... \leq p_n$, $p_j > 0$ (processing job j on machine i will take $\frac{p_j}{s_i}$ units of time, ie processors are related)
- Output : A schedule σ which assigns to each job one machine and one starting date
- Constraints :
  - A job can be processed on only one machine
  - No preemption, no restart
- Objective function : makespan

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Results on this problem

Existing results :

- $Q||Cmax$ is unary NP hard
- Gonzalez, Ibarra and Sahni [SIAM JoC 77] provided a 2 approximation algorithm (a list algorithm)
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Why is this interesting ?

- $\longrightarrow$ the algorithmic complexity of the $PTAS$ is huge ($O(n^{43})$ to get a ratio of $\frac{3}{2}$)
- $\longrightarrow$ our algorithm, based on dual approximation and list technique, is simple and quick
- $\longrightarrow$ a naive implementation of the Gonzalez algorithm leads to $O(nlog(n) + mn)$ while our algorithm is in $O(nlog(n) + mlog(m) + mnlog(\sum_{i=1}^{n} p_i))$

Layout

1. Two classical list algorithms applied on $Q||Cmax$
   - Analysis of EFT without sorting the tasks
   - Analysis of EFT with decreasing sort
2. Our $\frac{3}{2}$ approximation for $Q||C_{max}$
   - The algorithm
   - Proof and tightness of the bound

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Sum up of known results

- Applied on the $P||C_{max}$ problem
  - EFT is exactly the famous [Graham 66] algorithm
  - the approximation ratio is $2 - \frac{1}{m}$
- Applied on the $Q||C_{max}$ problem
  - the approximation ratio is $\theta(log(n))$ (Aspnes and al. [ACM 93])

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The Gonzalez ratio of 2

Before scheduling the last task n (the smallest) :

- let $Load_i$: total load of machine i before scheduling n
- completion time on machine i is $\frac{Load_i}{s_i}$
- let $W = \sum_{i=1}^{m} Load_i + p_n$ (total work)
- let $Q = \sum_{i=1}^{m} s_i$

Idea of the proof :

- $s_k C_{max} = Load_k + p_n$
  $s_i C_{max} \leq Load_i + p_n, \forall i \neq k$
- $\sum_{i=1}^{m} s_i \ C_{max} \leq \sum_{i=1}^{m} Load_i + mp_n$
  $QC_{max} \leq W + (m-1)p_n$
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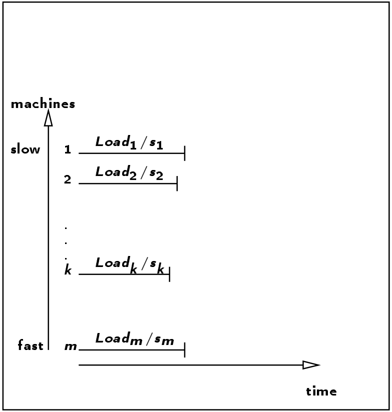
$C_{max} \leq 2\frac{W}{Q} \leq 2C^*_{max}$



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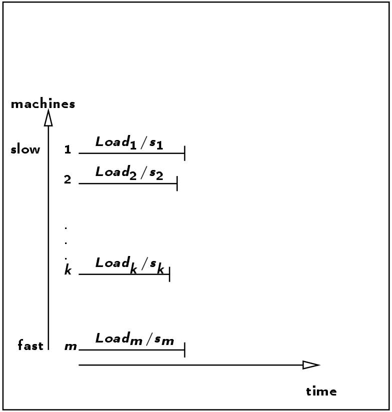

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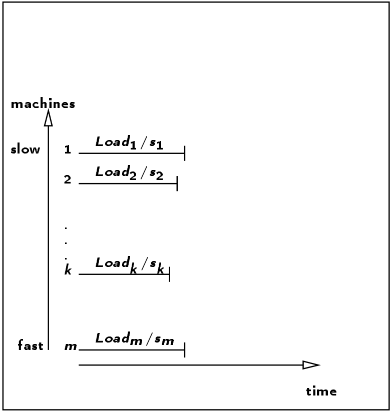

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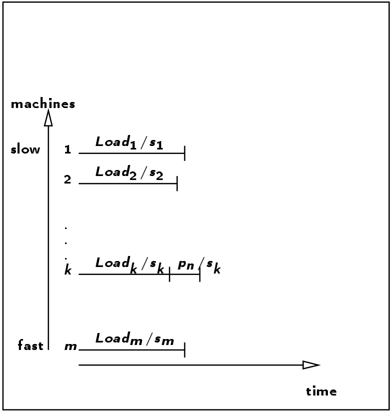

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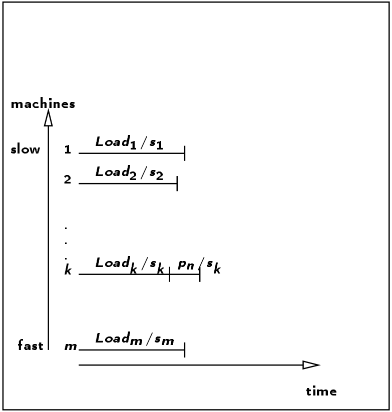

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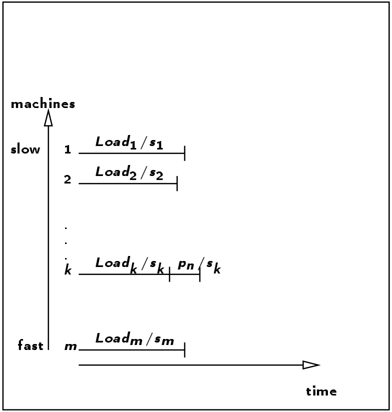

A worst case for EFT + Sort

Lower bound for the ratio

There exist instances for which $\frac{C_{max}}{C^*_{max}} \approx \frac{3}{2}$, for $m \longrightarrow \infty$

Let $s_1 = 1 \leq s_2 \leq \ldots \leq s_m$ and
$p'_1 \leq p_1 \leq p_2 \leq p_3, \ldots \leq p_m$, such that :

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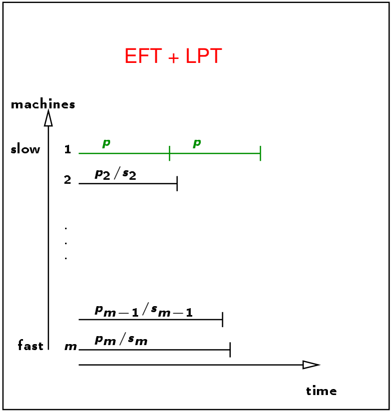

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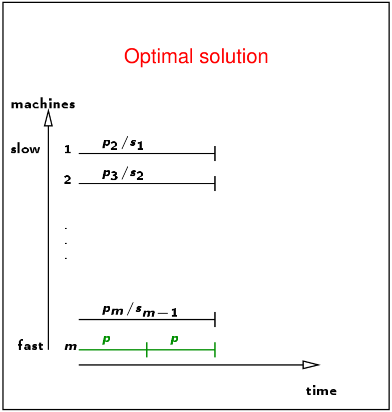

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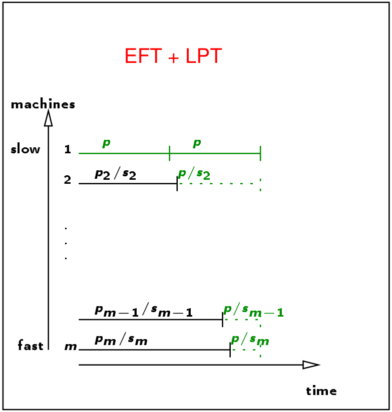

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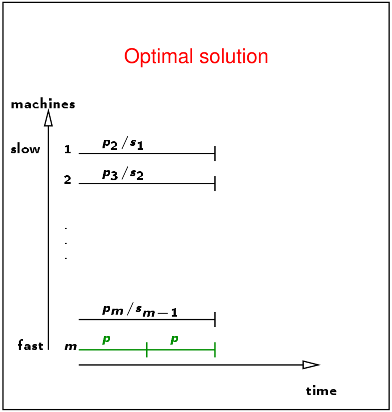

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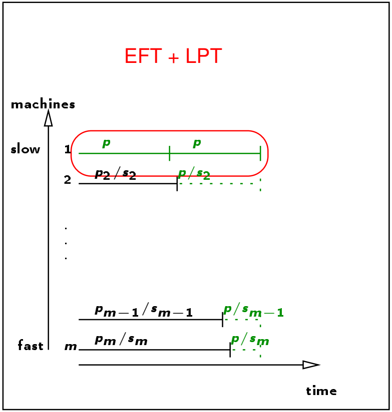

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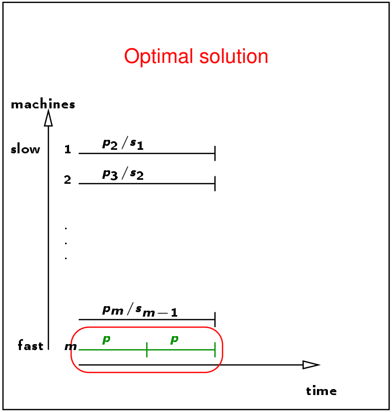

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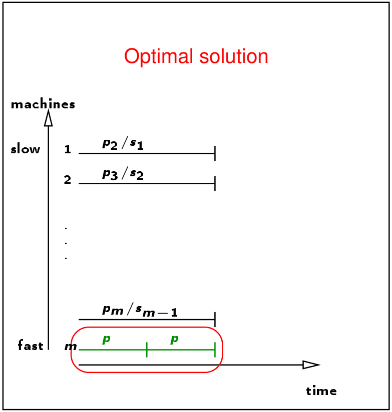

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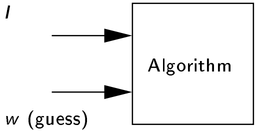
Presentation

- our algorithm uses the dual approximation technique :



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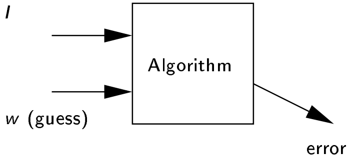

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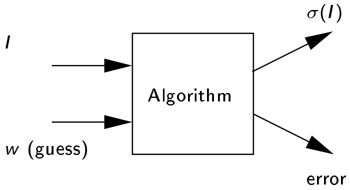

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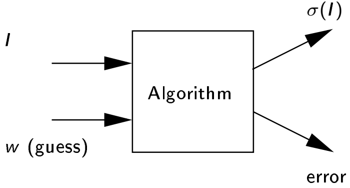

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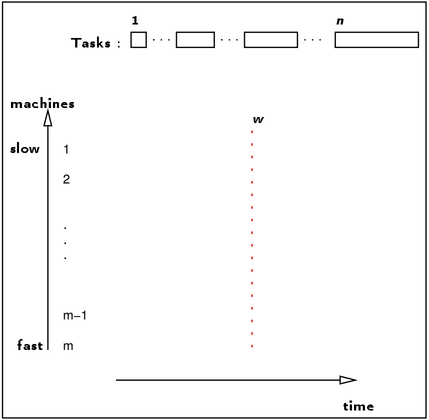

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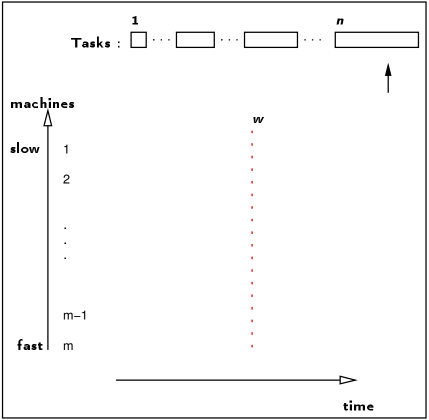

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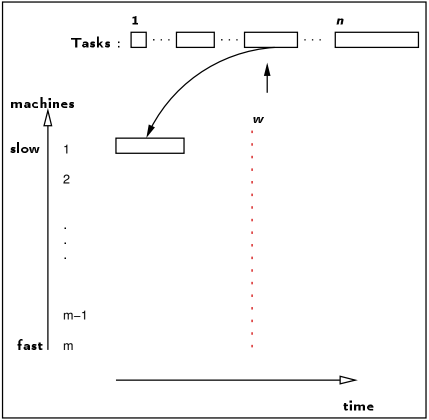

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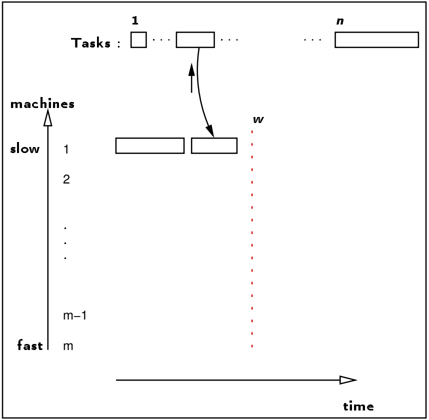

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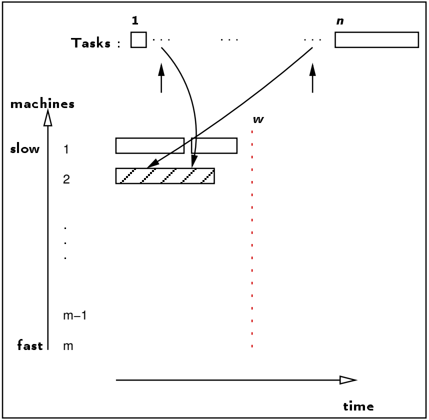

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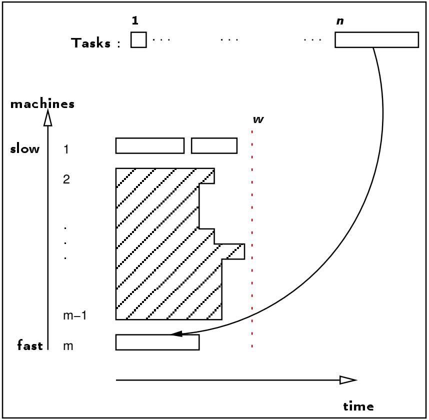

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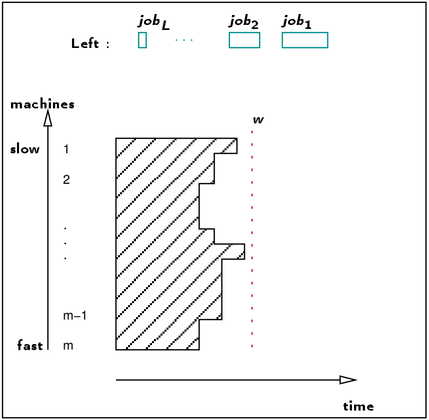

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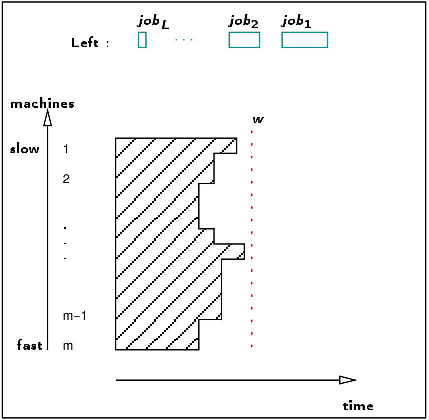

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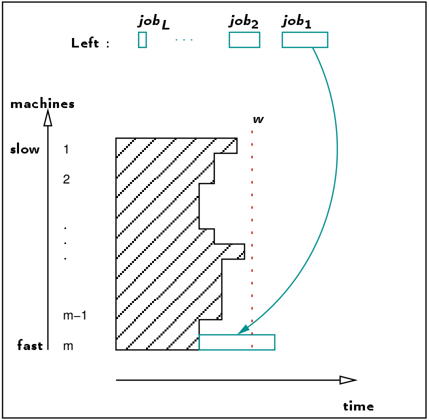

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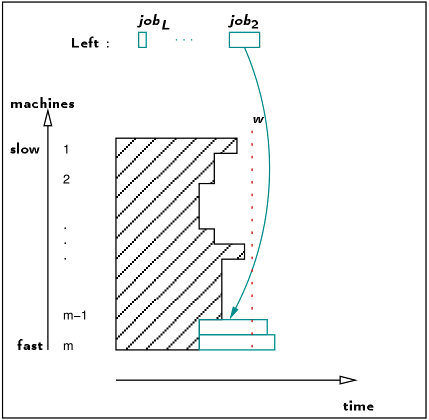

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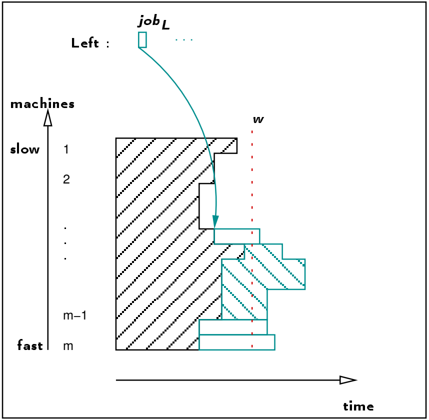
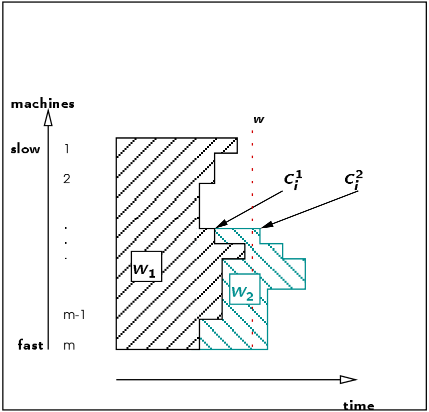


Notations

- let C_i^1 : completion time of machine i at the end of phase 1
- let C_i^2 : completion time of machine i at the end of phase 2
- $W_1 = \sum_{i=1}^{m} C_i^1 s_i$
- $W_2 = \sum_{j \in Left} p_{job_j}$
- $W = W_1 + W_2$



Proof of the 3/2 bound

Ratio of the algorithm

If the algorithm does not fail, $C_{max} \leq \frac{3}{2} w$

Proof :

- $L < m \implies$ all jobs are scheduled after phase 2
- $\forall i \in \{1, .., m - L\}, C_i^2 = C_i^1 \leq w$
- $\forall j \in \{1, .., L\}, \frac{job_j}{s_{m-j+1}} \leq \frac{w}{2} \implies$

$$\forall i \in \{m - L + 1, .., m\}, C_i^2 = C_i^1 + \frac{job_j}{s_{m-j+1}} \leq w + \frac{w}{2} = \frac{3}{2} w$$

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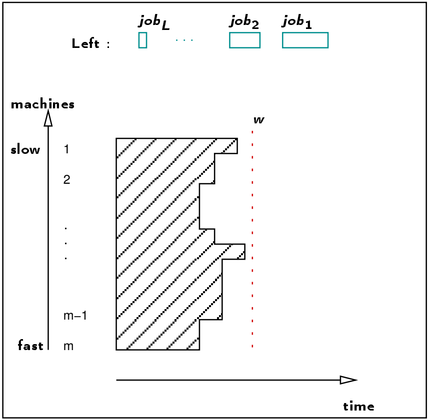

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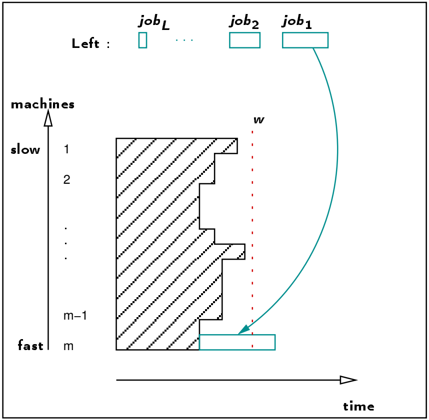

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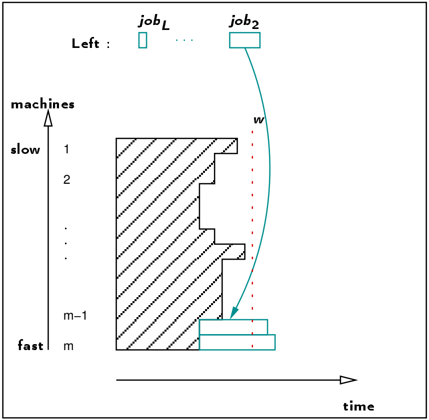

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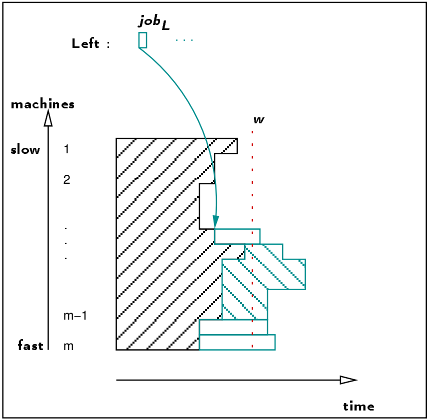

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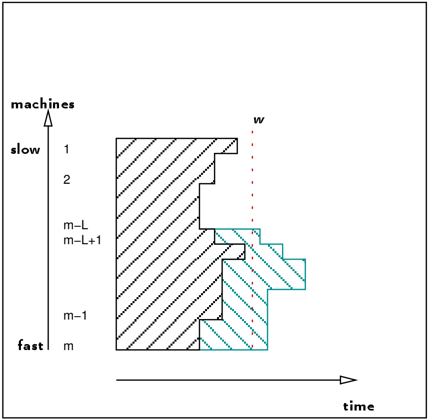

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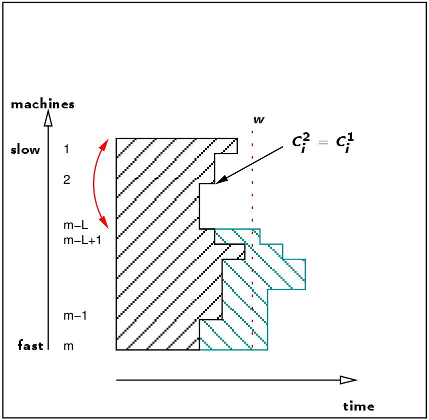

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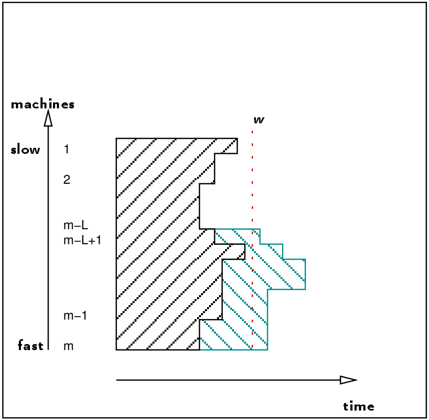

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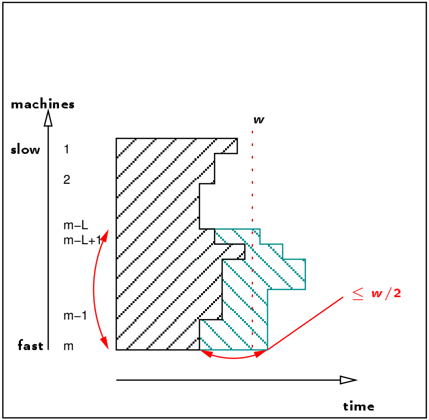

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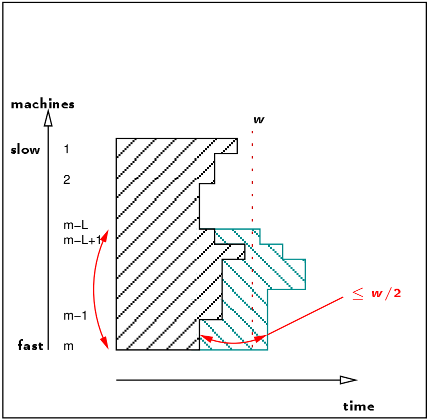

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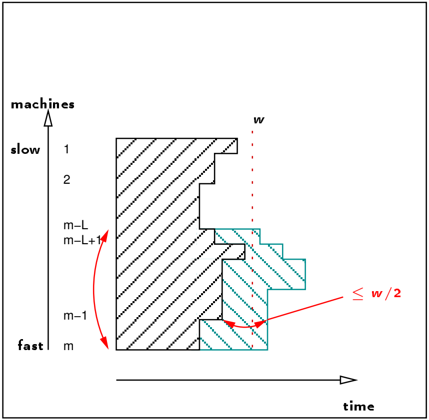

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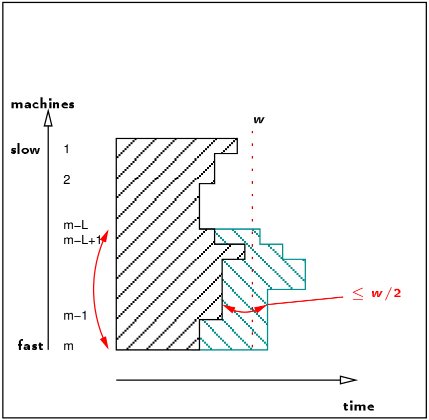

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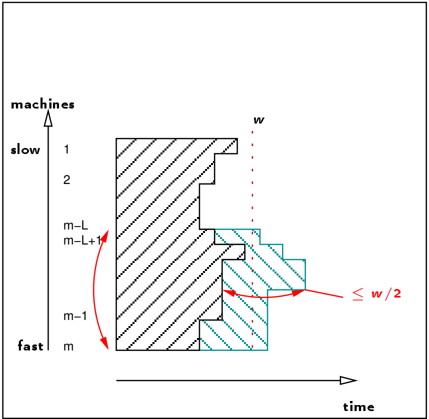

Proof of the 3/2 bound

Ratio of the algorithm

If the algorithm does not fail, $C_{max} \leq \frac{3}{2} w$

Proof :

- $L < m \implies$ all jobs are scheduled after phase 2
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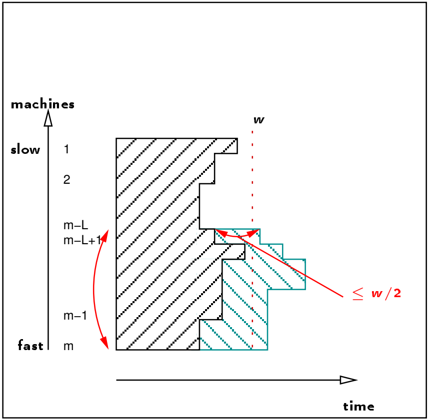

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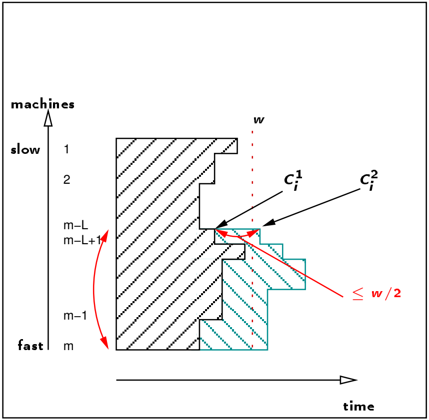

Proof of the first reject condition

First reject condition

$\neg(L < m) \Longrightarrow w < C^*_{max}$

Proof :

- By construction of *Left* :

  $\forall j \in Left, \forall i \in \{1, .., m\}, C_i^1 + \frac{job_j}{s_i} > w$ (1)

- Now suppose $L \geq m$

- using the classical Graham bound, we get $C^*_{max} \geq \frac{W}{\sum_{i=1}^{m} s_i} > w$

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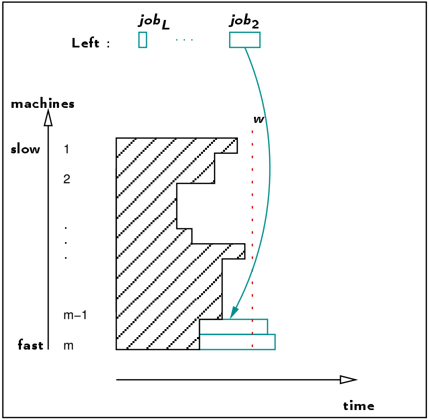

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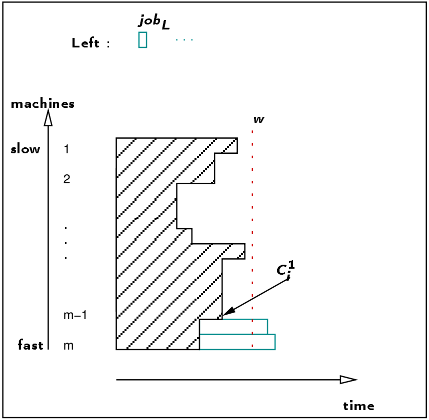

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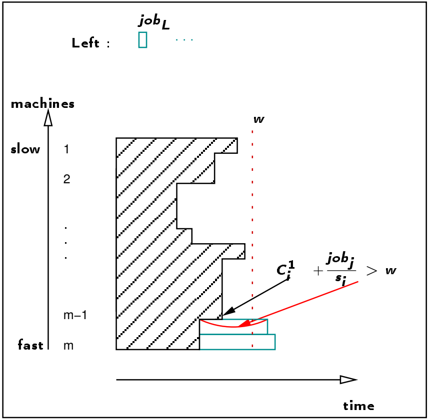

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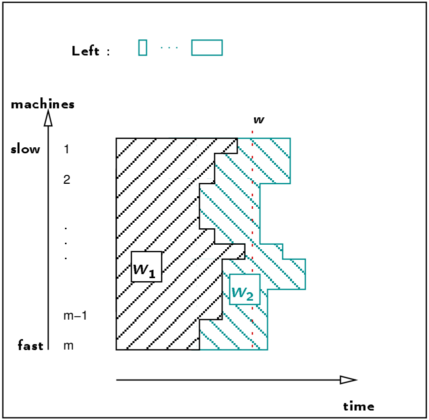

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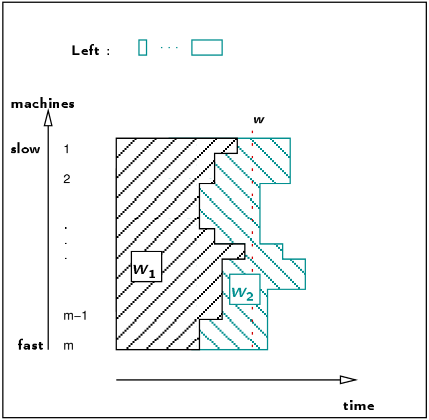

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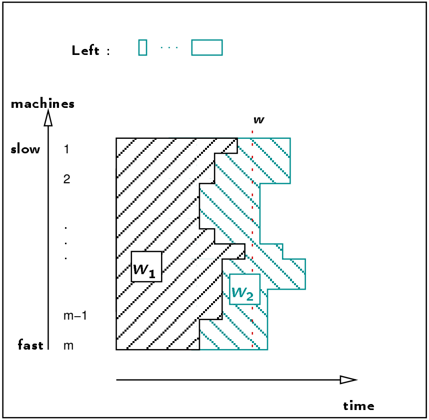

Proof of the second reject condition

Second reject condition

$$\neg(\forall j \in \{1,..,L\}, \frac{job_j}{s_{m-j+1}} \leq \frac{w}{2}) \Longrightarrow w < C^*_{max}$$

Proof :

- Suppose $\exists j_0, \frac{job_{j_0}}{s_{m-j_0+1}} > \frac{w}{2}$ (2) (let's take the first such j_0)
- Let $I' = \{j \in \{1,..,n\}, p_j \geq p_{job_{j_0}}\}$
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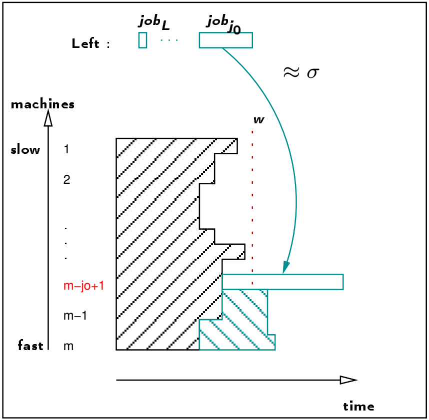

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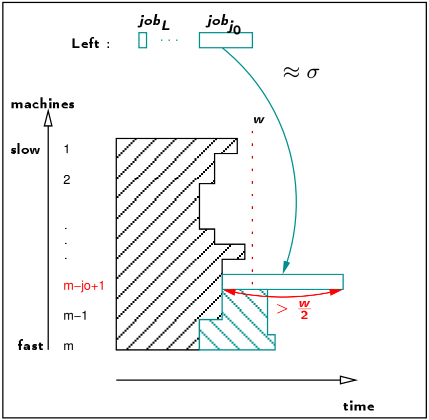

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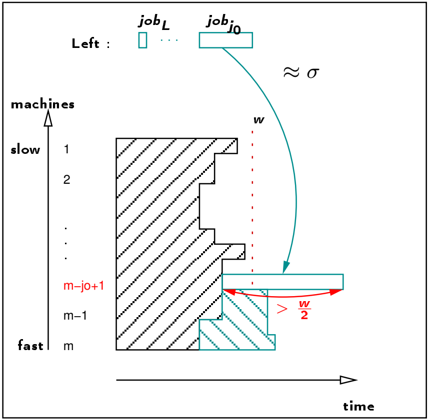

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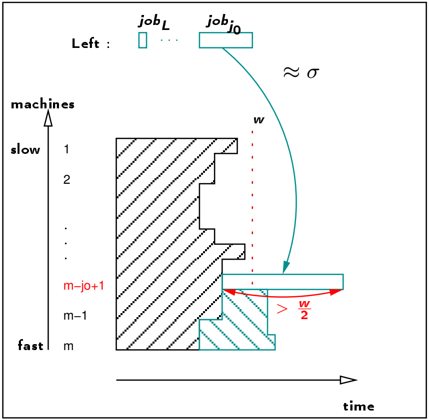

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To show that I' is impossible, we proved that :

- the total work W_{slow} scheduled on the "slow" machines is maximal (optimal)
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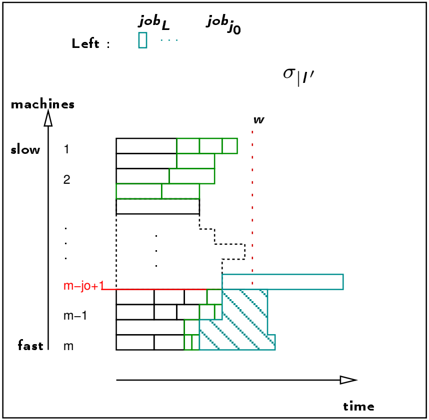

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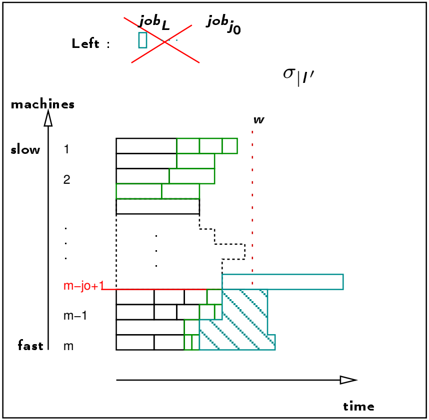

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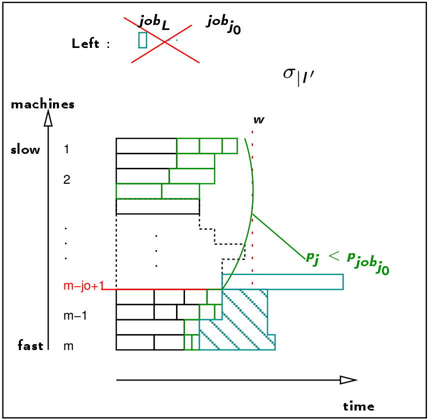

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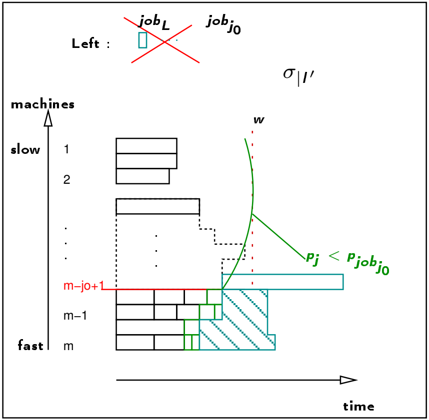

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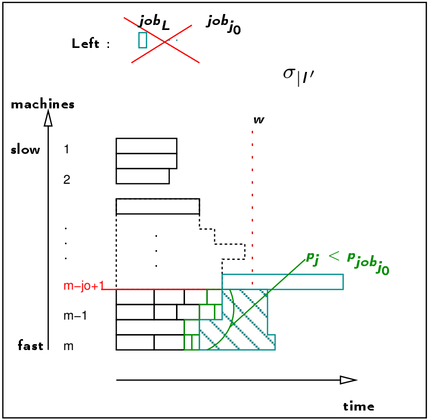

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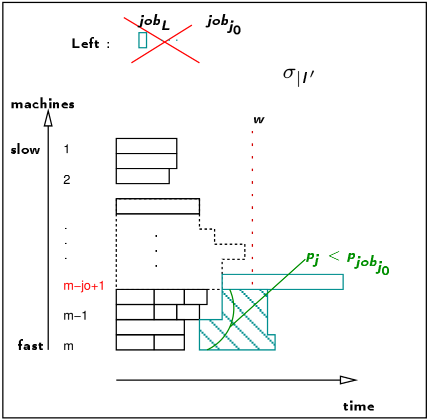

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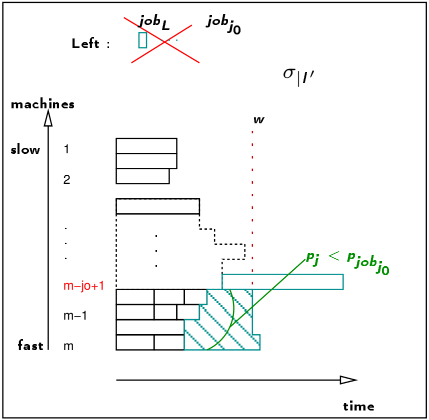

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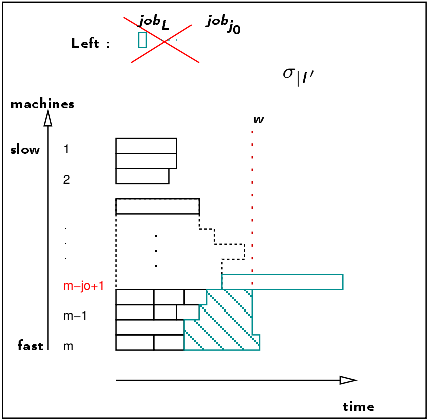

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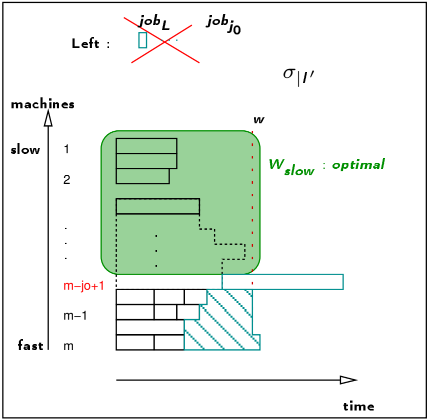

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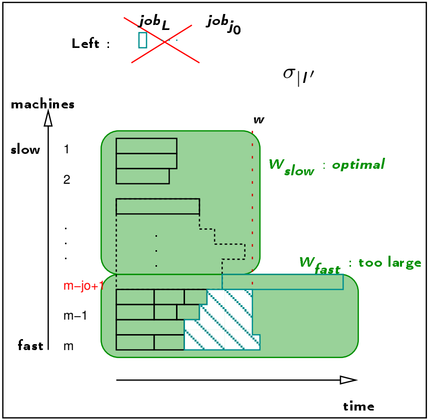
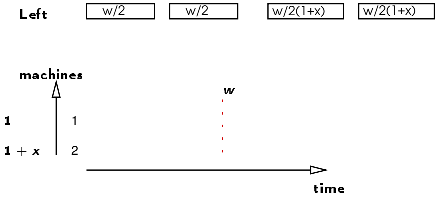


Tightness of the bound

Theorem

$\forall \epsilon > 0, \exists I / C_{max} \geq (\frac{3}{2} - \epsilon) C^*_{max}$



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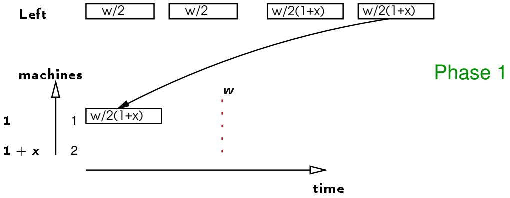

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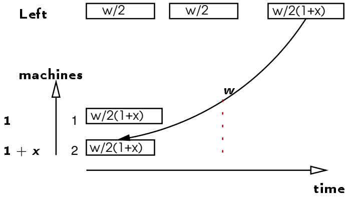

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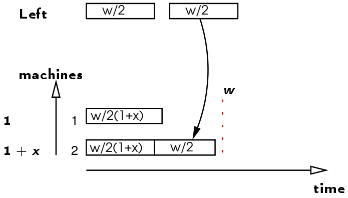

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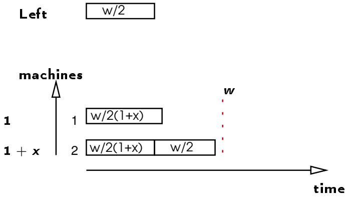

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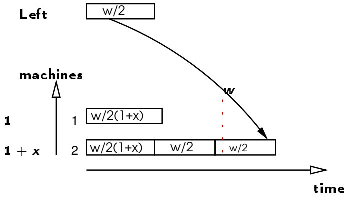

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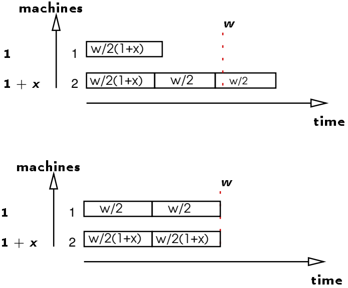
Left



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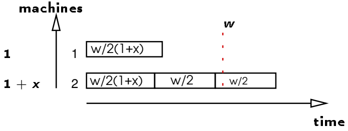
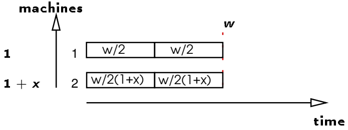
$\forall \epsilon > 0, \exists I / C_{max} \geq (\frac{3}{2} - \epsilon) C^*_{max}$

Left





$$\frac{C_{max}}{C^*_{max}} = \frac{(\frac{w}{2} + \frac{w}{1+x})}{w} = \frac{1}{2} + \frac{1}{1+x}$$

Notice that we could modify the phase 2 of our algorithm to avoid this case, but ...

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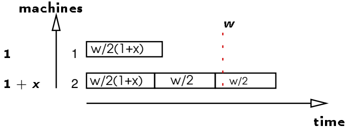

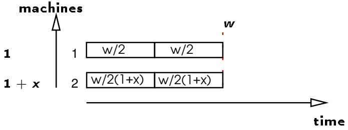

$$\frac{C_{max}}{C^*_{max}} = \frac{\left(\frac{w}{2} + \frac{w}{1+x}\right)}{w} = \frac{1}{2} + \frac{1}{1+x}$$

Notice that we could modify the phase 2 of our algorithm to avoid this case, but ...

Conclusion

For the problem $Q||C_{max}$,

- there is a fast $\frac{3}{2}$ approximation
- there is the PTAS of Hochbaum and Shmoys [SIAM JoC 88]

Thank you for your attention !