Using oracles for the design of efficient approximation algorithms

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We are interested here in oracle techniques for the design of approximation algorithms. Following the classical definition, an oracle is a black box capable of answering correctly and instantaneously any question. Several classical approximation scheme design techniques (typically $PTAS$) can be revisited using oracle.

Our objective in this work is to show that, conversely, oracle techniques are not limited to the design of $PTAS$. In particular, interactivity (using queries to oracle) may also lead to parameterized algorithm (whose complexity is exponential in a parameter, supposed to be "small"), that can be more practical than classical $PTAS$. Moreover, we aim at showing how it is possible to "degenerate" questions asked to the oracle to derive fast implementations of these interactive algorithms. These ideas will be illustrated on the classical makespan minimization on uniform machines problem ($Q||C_{max}$).

Context : interactive algorithms

Given an instance I of an optimization problem, an interactive algorithm A_{int} asks the oracle for a *guess*, in the form of a string $r_I^* \in R_I$, that generally provides some information on the structure of an optimal solution. Then, the algorithm constructs a solution $A_{int}(I, r_I^*)$ for the initial problem. From such an algorithm, it is possible to derive a "classical" algorithm A (without oracle), by either re-executing $A_{int}(I, r)$ for any $r \in R_I$, or constructing separately r_I^* (using another algorithm). Taking the example of scheduling problems, the asked information r_I^* is typically an "optimal configuration" of a well-chosen small subset of k tasks (among the n of the instance), like for instance the k biggest ones [5]. Such information may allow arbitrarily good approximation ratios (like $1 + \frac{1}{k}$) at the price of subset enumeration, when simulating the oracle.

Hence, it is clear that there exist deep connections between interactive algorithms and techniques for designing approximation schemes. As shown in [1], an interactive formulation allows natural alternative definitions of several classical techniques (as those presented in [10]). Most of such techniques are based on information obtained by exhaustive enumeration or by binary search. Replacing them by oracle answers separates difficulties due to the information determination from the ones due to its utilization.

Application on the classical $Q||C_{max}$ problem

Let us consider the problem of minimizing the makespan when scheduling independent tasks on uniform machines as a case study. It is shortly denoted by $Q||C_{max}$.

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Several approximation algorithms have been proposed for this problem. The 2 ratio (achieved by the classical Longest Processing Time algorithm [4]) has been improved to $\frac{3}{2}$ in [6] (using the dual approximation technique), and to 1.382-approximation in [2]. Among all existing approximation schemes, the most relevant here are the following (we list below the time complexity to achieve a ratio of $(1+\epsilon)$):

- $\mathcal{O}(mn^{\frac{10}{\epsilon^2}+3})$ in [6]
- $\mathcal{O}((\frac{1}{\epsilon}n^2)^{m-1})$ (also applies to $R||C_{max}$) in [7]
- $\mathcal{O}((n+1)^{\frac{m}{\epsilon}}poly(n,m))$ (also applies to $R||C_{max}$) in [9]
- $\mathcal{O}(n) + (\frac{log(m)}{\epsilon})^{\mathcal{O}(m^2)}$ (also applies to $R|c_{ij}|C_{max}$) in [3]
- $\mathcal{O}(2^{\mathcal{O}(1/\epsilon^2 log(1/\epsilon)^3)}poly(n,m))$ in [8]

We propose an interactive algorithm based on [6]. For any $a \in \mathbb{N}^*$, our algorithm guarantees an $1+\frac{1}{a}$ ratio by asking for each machine information on the "big" tasks (*i.e.* whose computation requires more than a fraction $\frac{1}{a}$ of the total computation time on this machine). Even if it is possible to derive from this algorithm an approximation scheme with the same complexity as the one in [9], we rather focus on the previously explained perspectives.

Firstly, we show how to reduce the amount of information asked, and thus the size of R_I, for small values of a (typically $a = 3$ or 4). We get for instance a $\frac{4}{3}$-approximation by only asking one bit of information for each machine, and the complexity of this algorithm (after simulating the oracle) is in $O(2^m poly(n,m))$. Thus, this $\frac{4}{3}$-approximation is practically faster than the better approximation schemes applied for $\epsilon = \frac{1}{3}$ (as there is no constant hidden in the exponent).

Secondly, we discuss efficient implementations where the algorithms avoid asking some sub-parts of the question. The key idea is to check (in $\mathcal{O}(1)$) if some additional *a priori* unexpected conditions become true during the execution on the particular current instance, allowing then to make a local optimal decision (without oracle query).

The approach presented in this work leads to the following natural questions for $Q||C_{max}$. They will be discussed during the presentation.

- Using only one bit of information for each machine, what information should be asked to obtain a ratio better than $\frac{4}{3}$?
- How to reduce the amount of information used for larger values of a ?

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