Efficient polynomial L_∞-approximations

ARITH 18 - Montpellier

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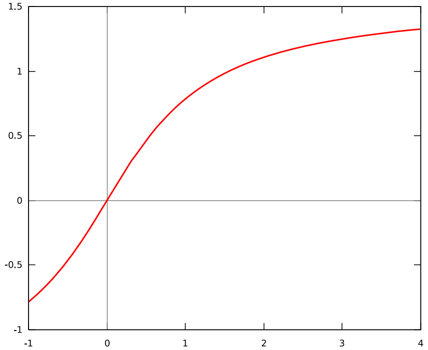
Functions approximation

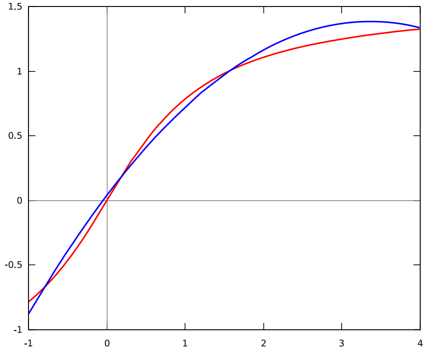


Graph of $f : x \mapsto \arctan(x)$

(interval $[-1, 4]$)

- Let f be a real valued function : $f : [a, b] \longrightarrow \mathbb{R}$.

Functions approximation



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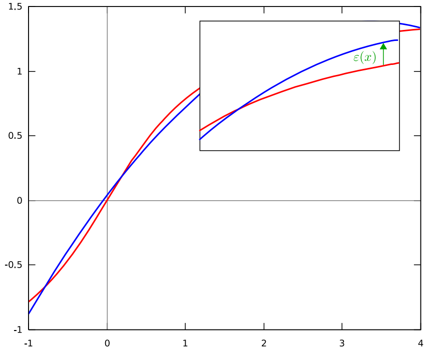

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- Let f be a real valued function : $f : [a, b] \longrightarrow \mathbb{R}$.
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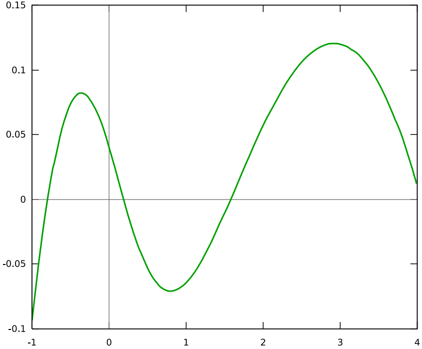
- Approximation error at point x: $\varepsilon(x) = p(x) - f(x)$.

Approximation error



- $\varepsilon(x) = p(x) - f(x)$ over $[a,\ b]$

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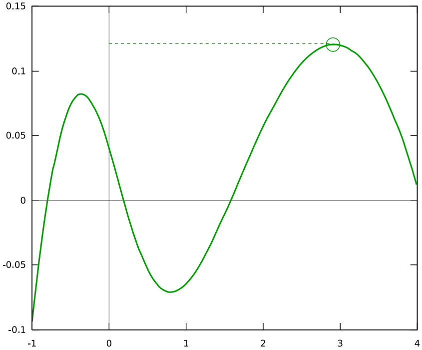

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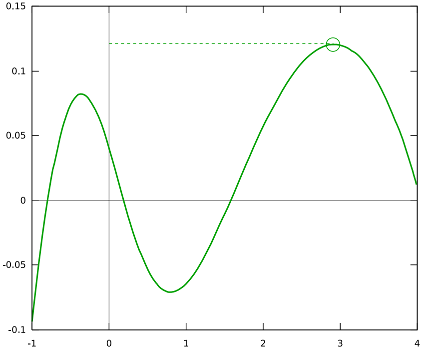

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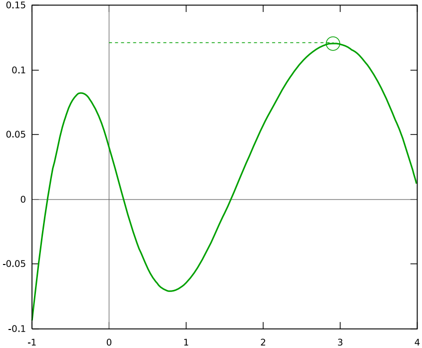

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- Infinite norm: $\|\varepsilon\|_\infty = \|p - f\|_\infty = \max\{|\varepsilon(x)|,\ x \in [a,\ b]\}$
- Best approximation problem:

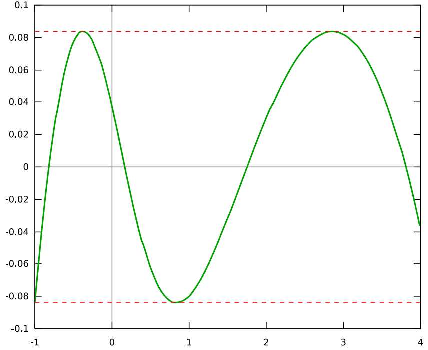
  given a degree n, find $p \in \mathbb{R}_n[X]$ minimizing $\|p - f\|_\infty$.

Theory of polynomial approximation

Facts:

- There exists a unique best approximation polynomial.

Theory of polynomial approximation



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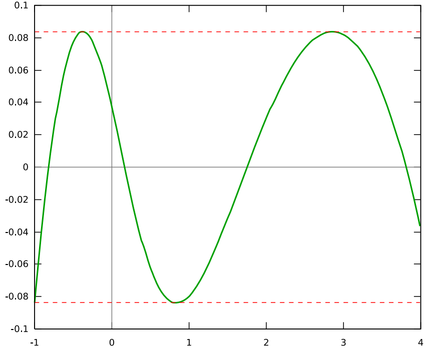

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- There exists a unique best approximation polynomial.
- Characterization: Chebyshev's theorem.
- To compute it: Remez' algorithm (`minimax` in Maple).

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- In practice: one has to store the coefficients into floating-point numbers.
- Naive method: compute the minimax with Remez' algorithm and a high precision. Then round each coefficient to the nearest floating-point number.

Failure of the naive method

- Example with $f(x) = \log_2(1 + 2^{-x})$:
  - on $[0; 1]$
  - approximated by a degree 6 polynomial
  - with single precision coefficients (24 bits).

Minimax	Naive method	Optimal
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- The problem has been studied by
  - W. Kahan;
  - D. Kodek (precision $t < 10$, degree $n < 20$);
  - N. Brisebarre, J.-M. Muller and A. Tisserand (using linear programming).

Description of our method

Our goal: find p approximating f with the following form:

$$m_0 \cdot 2^{e_0} + m_1 \cdot 2^{e_1} X + \cdots + m_n \cdot 2^{e_n} X^n \qquad (m_i \in \mathbb{Z}).$$

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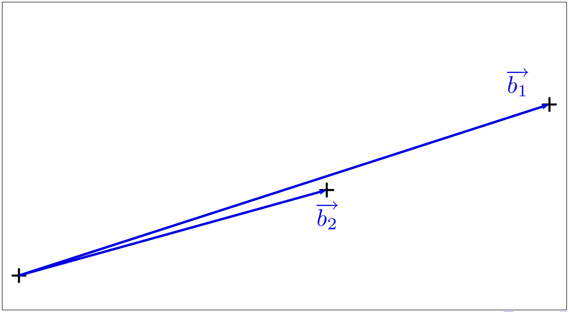
  - Rewritten with vectors:

$$\underbrace{m_0 \begin{pmatrix} 2^{e_0} \\ 2^{e_0} \\ \vdots \\ 2^{e_0} \end{pmatrix} + \cdots + m_n \begin{pmatrix} 2^{e_n} \cdot x_0^n \\ 2^{e_n} \cdot x_1^n \\ \vdots \\ 2^{e_n} \cdot x_n^n \end{pmatrix}}_{\Gamma \text{ of the form } \mathbb{Z}\overrightarrow{b_0} + \mathbb{Z}\overrightarrow{b_1} + \cdots + \mathbb{Z}\overrightarrow{b_n}} \simeq \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\overrightarrow{t} \in \mathbb{R}^{n+1}} \quad .$$

Notions about lattices

Let $(\overrightarrow{b_1}, \cdots, \overrightarrow{b_n})$ be a basis of a real vector space.



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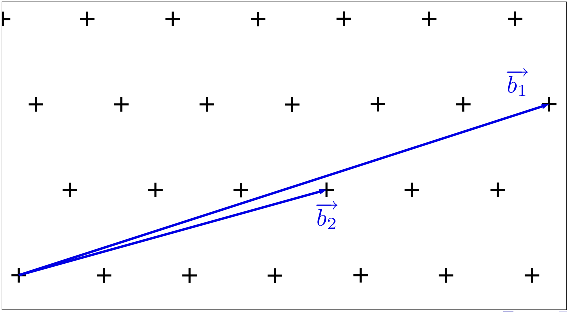
$$\Gamma = \mathbb{Z}\overrightarrow{b_1} + \mathbb{Z}\overrightarrow{b_2} + \cdots + \mathbb{Z}\overrightarrow{b_n} \quad .$$



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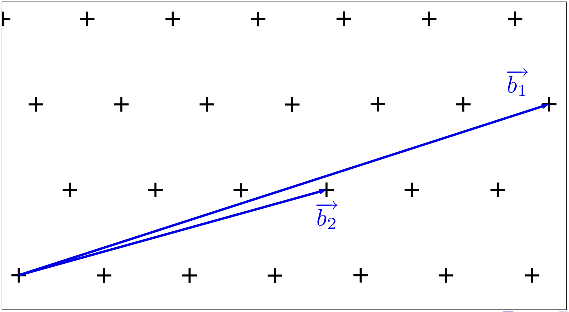

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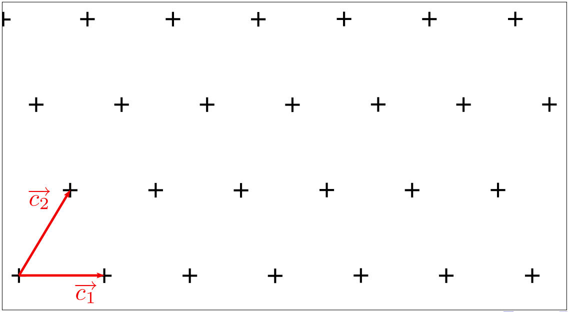

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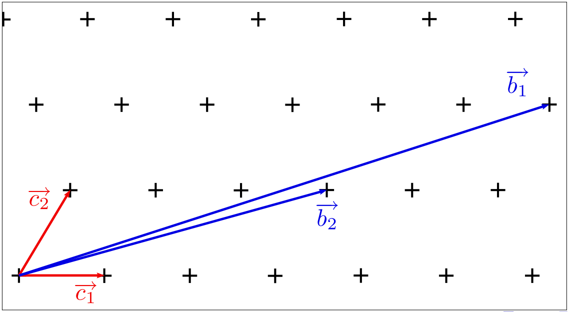

Notions about lattices

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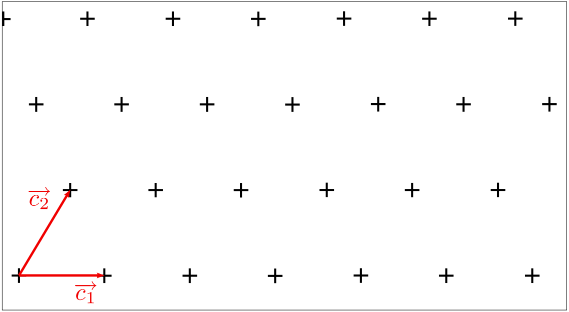
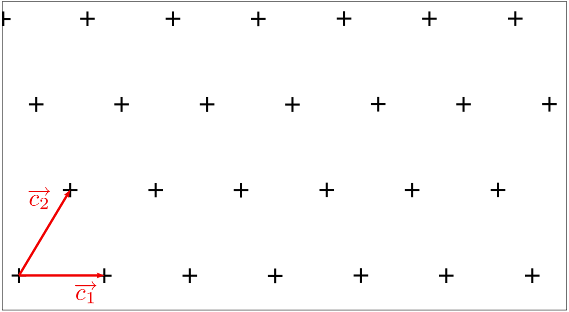


Notions about lattices

Algorithmic problems:

- Shortest vector problem (SVP)



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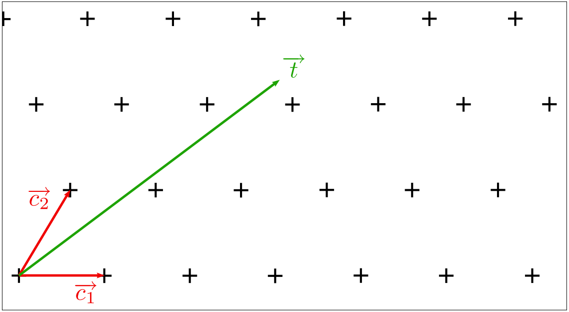

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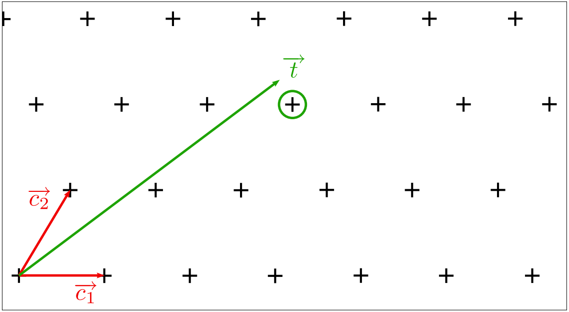

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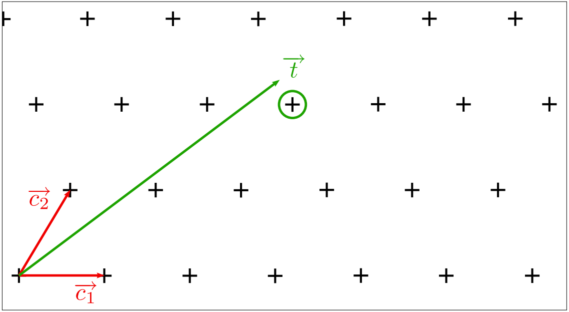
LLL algorithm: Lenstra, Lenstra Jr. and Lovász.



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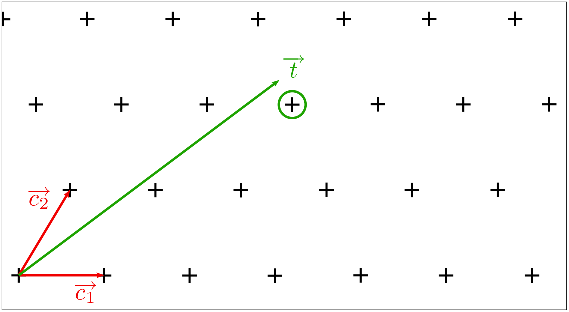
LLL algorithm: finds pretty short vectors in polynomial time.



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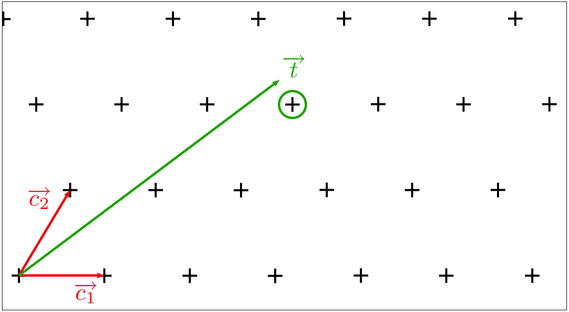
LLL algorithm: used by Babai to solve an approximation of CVP.



A concrete and toy case

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- We want to approximate $f : x \mapsto \log_2(1 + 2^{(-x)})$
  - on $[0, 1]$
  - by a polynomial of degree 6.
  - Each coefficient is stored in a single-precision number (24 bits).

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Best real	Naive method	Enhanced method
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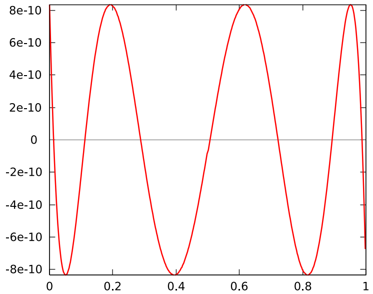

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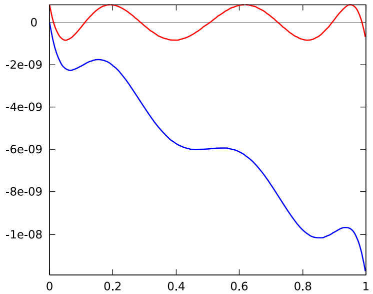

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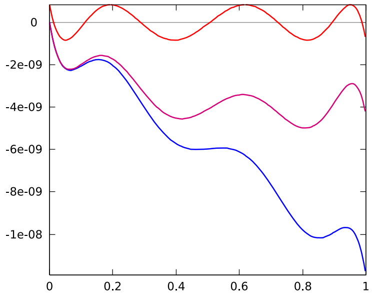

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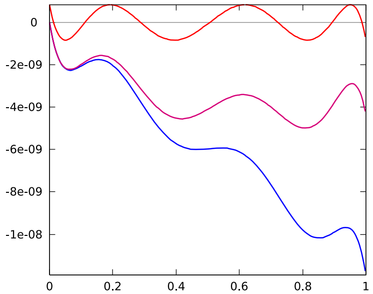

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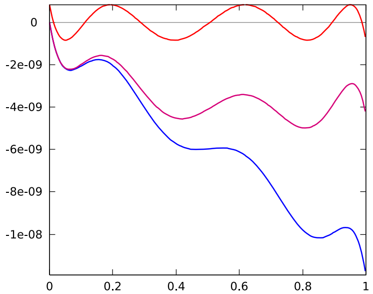

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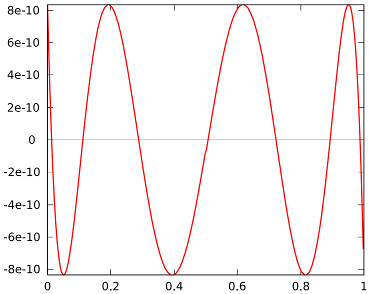

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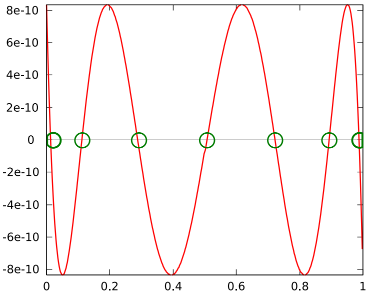

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- Chebyshev's theorem gives $n+1$ such points.

Results with our method

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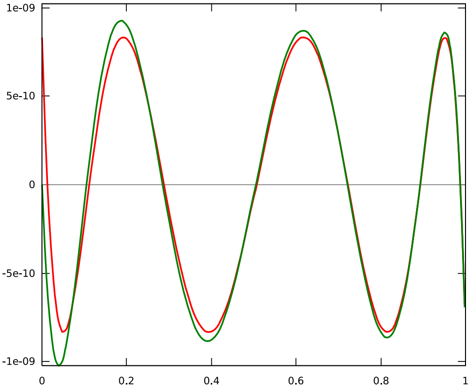

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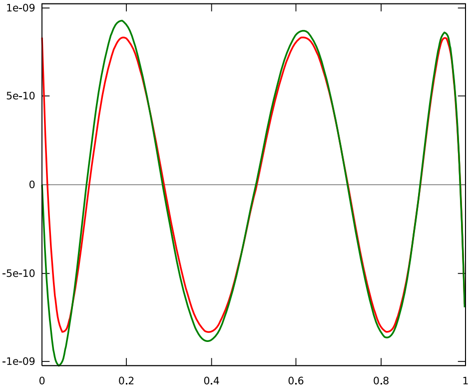

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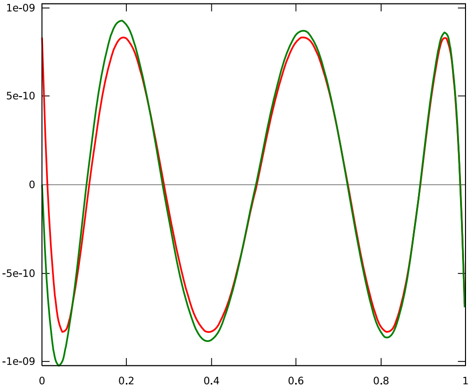

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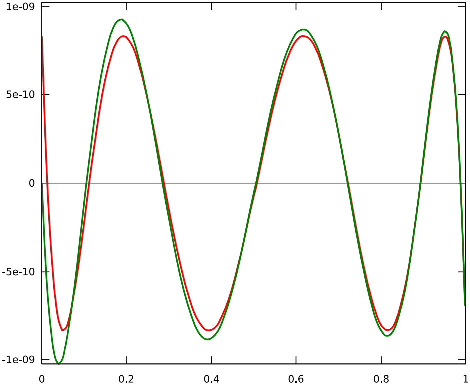

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- The definition often gives a natural way to find a polynomial approximation of f.
  $\hookrightarrow$ for instance: a truncated power series with a formally computed bound on the error.
- Truncated power series are useful but…
  …usually inefficient in term of number of operations.
- Example: $\exp(x)$ on $[-1; 2]$ with an absolute error ≤ 0.01:
  - the series must be truncated to a degree 7 polynomial;
  - a degree 4 polynomial is sufficient.

Chebyshev's theorem

Theorem (Chebyshev)

Let f be a continuous function on $[a, b]$. Let $\mu = \inf\{\|f - p\|_\infty\}_{p \in \mathbb{R}_n[X]}$. Then, p satisfies $\|f - p\|_\infty = \mu$ if and only if there exist $n + 2$ points

$$x_0 < x_1 < \cdots < x_{n+1}$$

in $[a, b]$ such that

1. $\forall i \in [\![0, n+1]\!], |f(x_i) - p(x_i)| = \|f - p\|_\infty$
2. *For all $i \in [\![0, n]\!]$, the signs of $f(x_{i+1}) - p(x_{i+1})$ and $f(x_i) - p(x_i)$ are different.*