Autour de la conjecture d'Hadwiger

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Introduction

k-coloration : $\phi : V(G) \rightarrow \{1, \ldots, k\}$, such that $\forall uv \in E$, $\phi(u) \neq \phi(v)$.

minor : edge contraction, edge deletion, vertex deletion.

Hadwiger's conjecture

Conjecture (Hadwiger)

Every K_k minor-free graph is $(k-1)$-colorable.

True for $k \leq 6$ [Dirac, Wagner, Appel & Haken, RST]
Open for $k \geq 7$.

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Every K_7 and $K_{4,4}$ minor-free graph is 6-colorable.

Let's look for a minimal counter-example G, i.e a 7-chromatic K_7 minor-free graph.

Theorem (Mader, 1968)

G is 7-connected.

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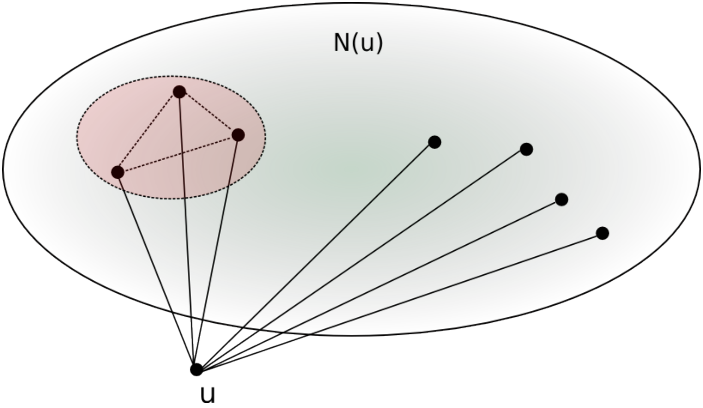
Kawarabayashi & Toft theorem

Lemma

There is no stable of size at least 3 *in the neighbourhood of a vertex of degree* 7*.*

Proof : Stables



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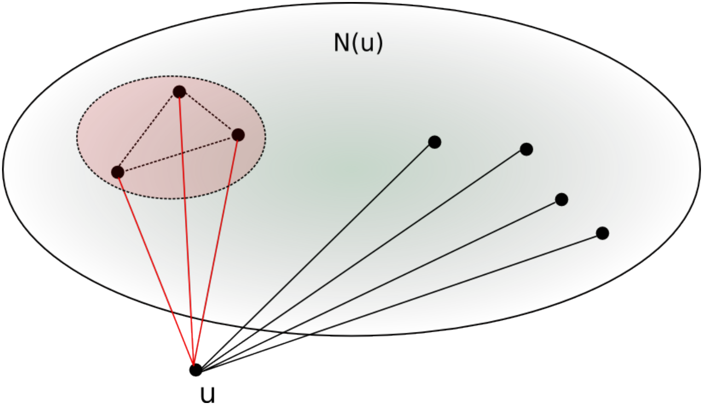

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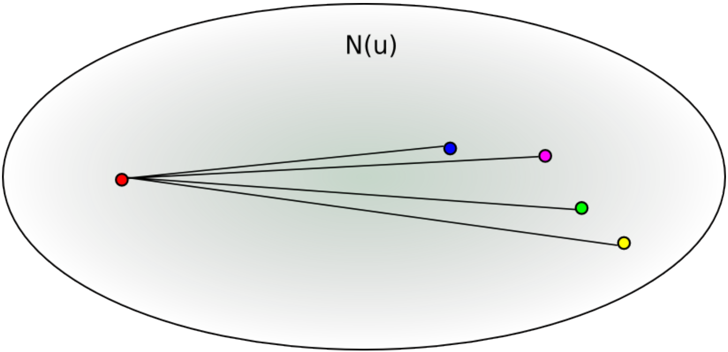

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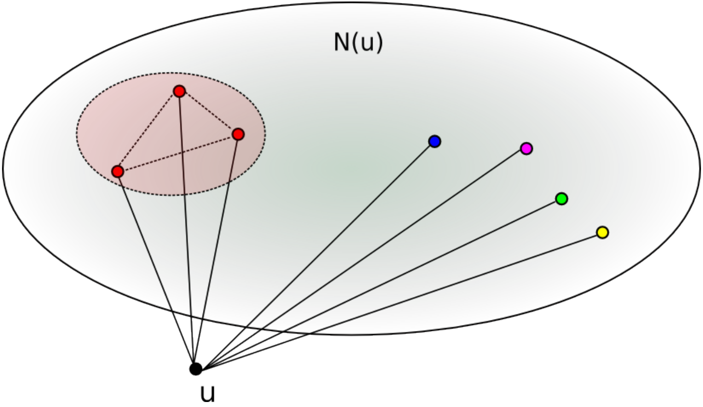

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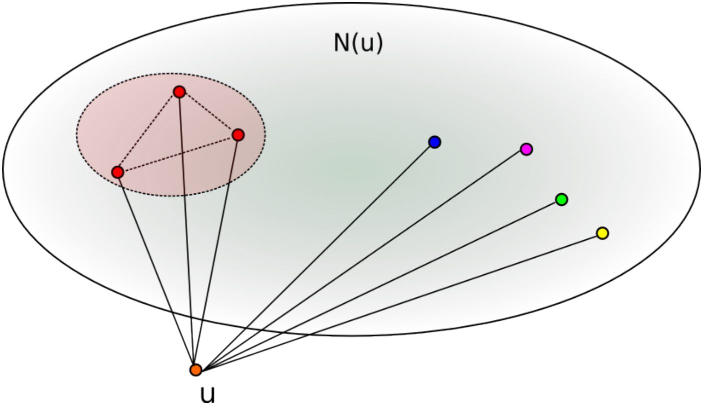

Proof : Neighbourhood

Question : What does the neighborhood of a vertex of degree 7 look like?

- If it is not connected : every component is a clique (only one possibility)
- If it is connected, it's obtained by taking a 5-cycle and replace vertices by cliques (3 possibilities)

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Proof : Finding vertices of degree 7

Theorem (Jørgensen, 2001)

Every 4-connected $K_{4,4}$ minor-free graph has at most $4n - 8$ edges.

- There are at least 16 vertices of degree 7
- Every neighbourhood contains a K_4

=> There is a least 4 different K_5 subgraphs

=> Combine them to create a K_7 or $K_{4,4}$. □

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Open Problems

Question (Kawarabayashi & Toft, 2005)

Does every K_7 and $K_{4,5}$ (resp. $K_{3,6}$) minor-free graph is 6-colorable.

Question

Can we found vertices of degree less than 9 in a K_7 and $K_{4,5}$ (resp. $K_{3,6}$) minor-free graph ?

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How many edge has a $K_{4,5}$ (resp. $K_{3,6}$) minor-free graph ?

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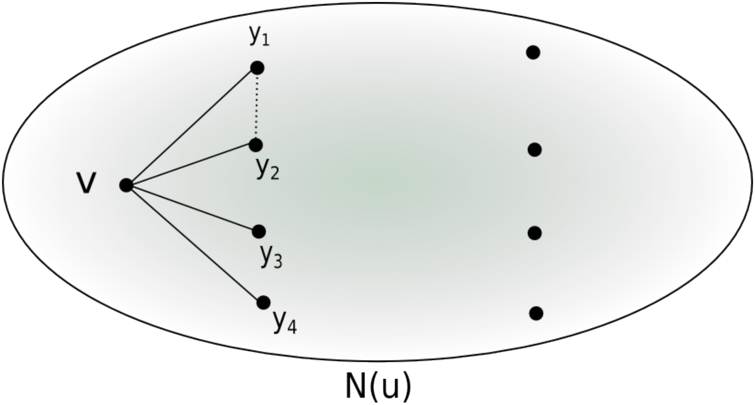
Lemma

- $\deg(u) = 9$
- *$G[N(u)]$ does not contains K_5 has a subgraph*
- $\deg_{N(u)}(v) > 4$

Proof



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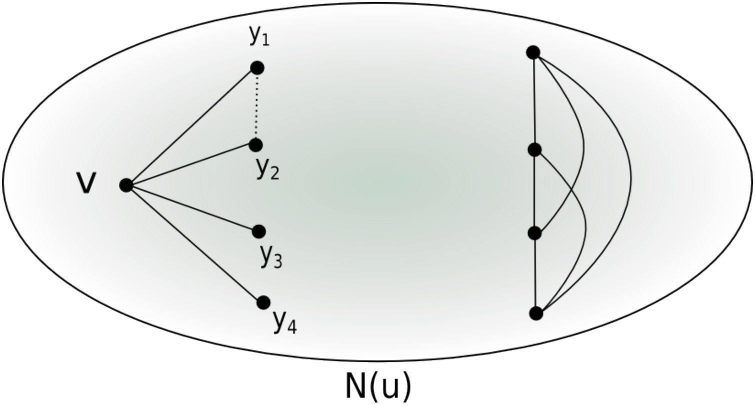

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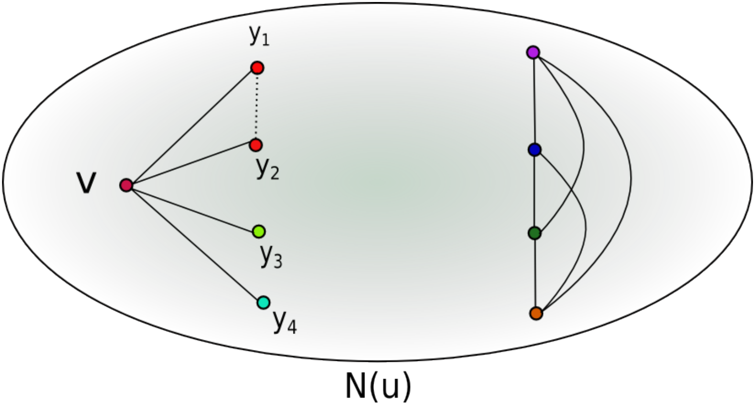

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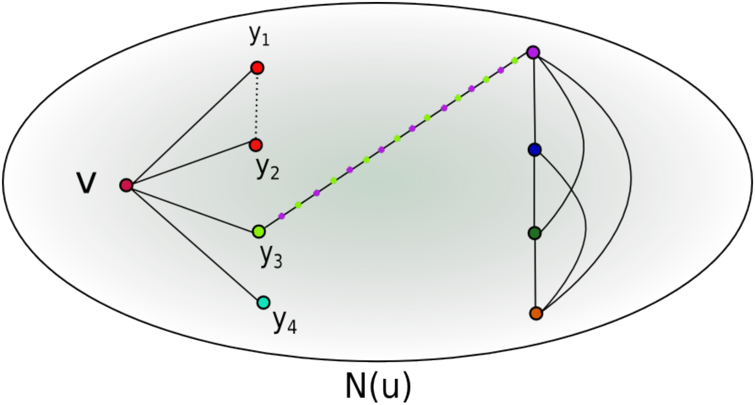

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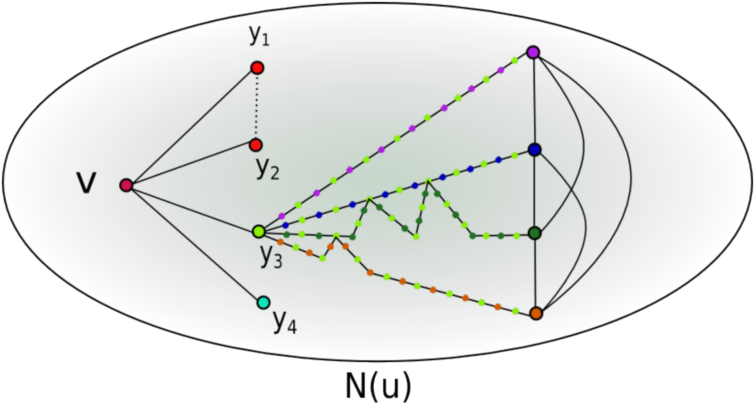

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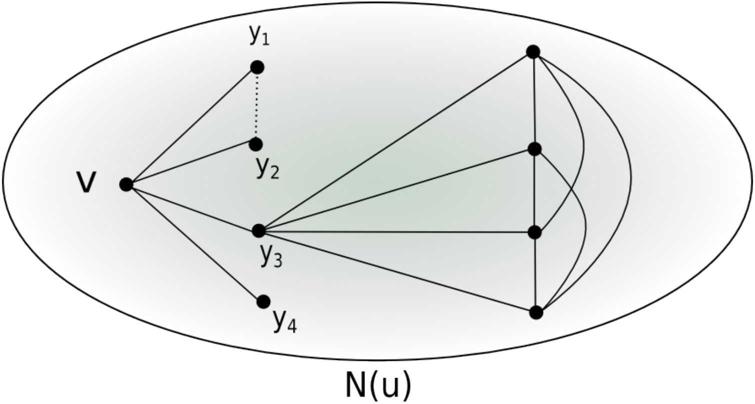

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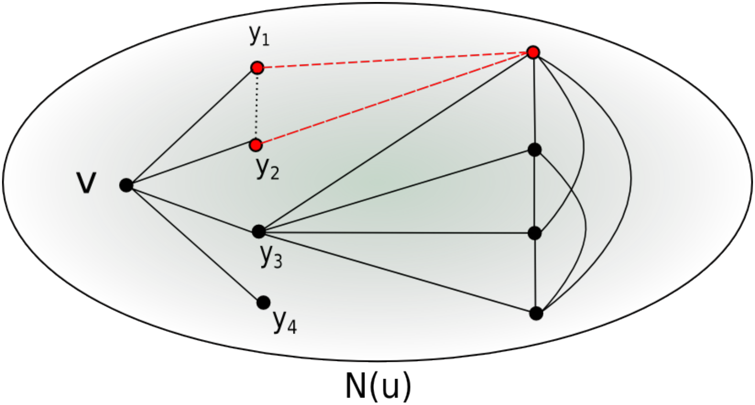

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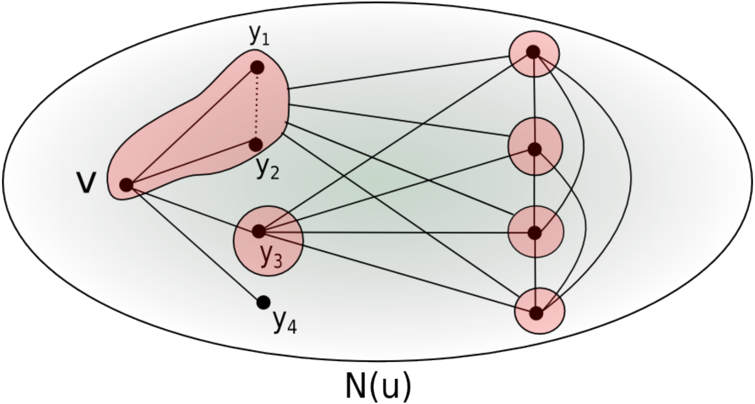

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Contradiction. □

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How to prove that K_7 (resp. K_8) minor-free graphs are 7- (resp. 9-) colorable ?

The neighbourhood of the vertices of degree 9 can be "sparse", e.g. three disjoint triangles.

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Double-critical Graph Conjecture

Definition

A connected k-chromatic graph G is double-critical if for all edges uv, the graphs $G - u - v$ is $(k-2)$-colorable.

Conjecture (Erdős & Lovasz, 1966)

K_k are the only k-chromatic double-critical graphs.

True for $k \leq 5$ [Mozhan, Stiebitz], open for $k \geq 6$.

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Double-critical Graph Conjecture

Theorem (Basic properties)

Let $G \neq K_k$ be a k-chromatic double-critical graph. The following properties holds :

- *G does not contain K_{k-1} as a subgraph,*
- *G has minimum degree at least $k+1$,*
- *Each edge of G belongs to at least $k-2$ triangles.*

Double-critical Hadwiger Conjecture

Conjecture (Karawabayashi, Toft & Pedersen, 2010)

Every k-chromatic double-critical graphs contains K_k as a minor.

- True for $k \leq 7$ [Kawarabayashi, Toft & Pedersen, 2010]
- Every 8-chromatic double-critical graphs contains K_8^- as a minor [Pedersen, 2011]
- True for $k = 8$ [A. & Gonçalves, 2012]
- Open for $k \geq 9$

Theorem (A. & Gonçalves, 2012)

If G has an edge and each edge belongs to at least 5 (resp. 6) triangles, then G has a K_7 (resp. K_8 or $K_{2,2,2,2,2}$) minor.

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Every 6-chromatic double-critical graph contains K_5 as a subgraph.

Question (Kriesell)

Does every 6-chromatic double-critical graph contains K_4 as a subgraph ?

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