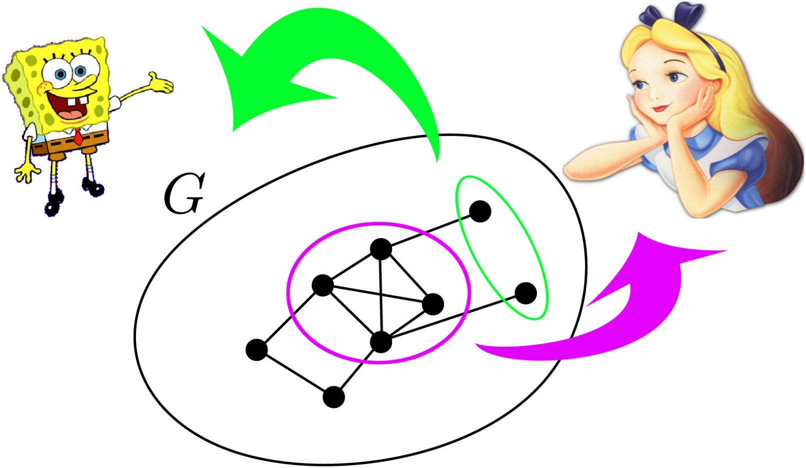
Separation of cliques and stable sets

Nicolas Bousquet Aurélie Lagoutte Stéphan Thomassé

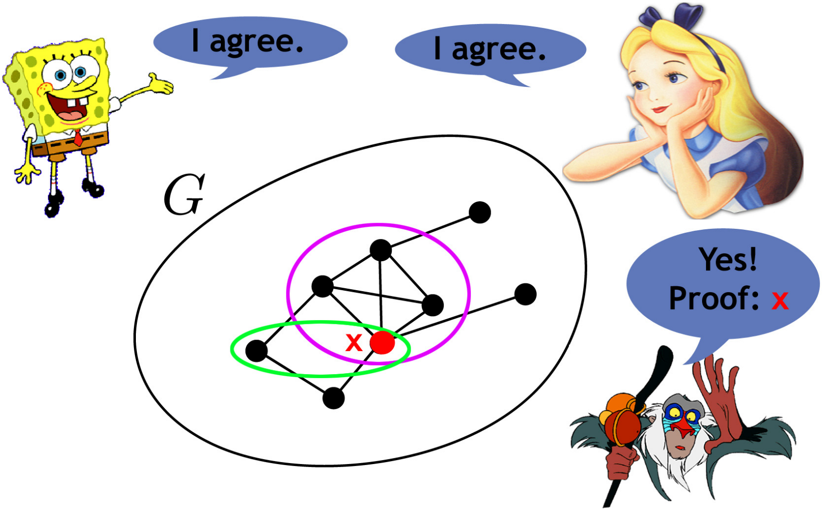
Séminaire AlGCo [1]

1. Slides by Nicolas Bousquet and Aurélie Lagoutte

Clique vs Independent Set Problem



Clique vs Independent Set Problem : Non-det. version



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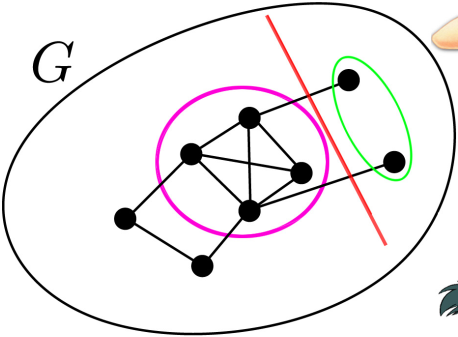

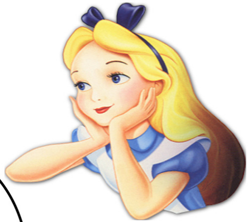

Clique vs Independent Set Problem

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Find a *CS-separator* : a family of cuts separating all the pairs Clique-Stable set.

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Non-deterministic communication complexity $= \log m$
where m is the minimal size of a CS-separator.
If $m = n^c$, then complexity$=\mathcal{O}(\log n)$.

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Idea : Covering the Clique - Stable Set matrix with monochromatic rectangles.

CL-IS problem : Bounds

Upper bound

There is a Clique-Stable separator of size $\mathcal{O}(n^{\log n})$.

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Question

Does there exists for all graph G on n vertices a CS-separator of size poly(n) ?

Extended formulations : Definitions

Stable set polytope

- n dimensionnal space.
- Characteristic vector of S : $\chi_v^S = 1$ if $v \in S$.
- Number of constraints needed to define this polytope ?

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Extented formulation

Free to increase the dimension, what is the minimum number of half-spaces necessary to define the polytope ?

Reformulation

Free to add new variables, what is the minimum number of constraints needed to find the set of solutions ?

Extended formulations and CL-IS problem

Implication (Yannakakis '91)

If the Stable Set polytope has a polynomial extended formulation, then the Clique vs Stable Problem has a $\mathcal{O}(\log n)$ solution.

Extended formulations and CL-IS problem

Implication (Yannakakis '91)

If the Stable Set polytope has a polynomial extended formulation, then the Clique vs Stable Problem has a $\mathcal{O}(\log n)$ solution.

$\Rightarrow$ Fiorini et al. (2012) disprove the existence of such an extended formulation for the stable set polytope.

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There is a $\mathcal{O}(n^{5+\epsilon})$ CS-separator for random graphs.

Proof :
Let p be the probability of an edge. $\Rightarrow$ Draw randomly a partition (A, B).
A vertex v is in A with probability p and is in B otherwise.
$\Rightarrow$ Draw $\mathcal{O}(n^{5+\epsilon})$ such partitions.
W.h.p. there is a partition which separates C, S.

Split-free graphs

Theorem

Let H be a split graph. There is a polynomial CS-separator for H-free graphs.

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Idea : $\mathcal{O}(|H|)$ vertices of the clique "simulate" the pair C,S.

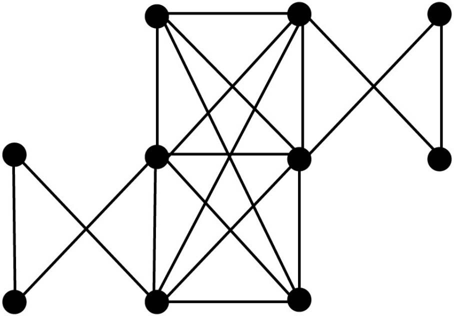
1. Clique-Stable set separation
   - CL-IS problem
   - Extended formulations
   - Some classes of graphs

2. Alon-Saks-Seymour Conjecture
   - A generalization of Graham-Pollack
   - Equivalence theorem

3. Constraint satisfaction problem

4. Prospects

Bipartite packing $\mathrm{bp}(G)$



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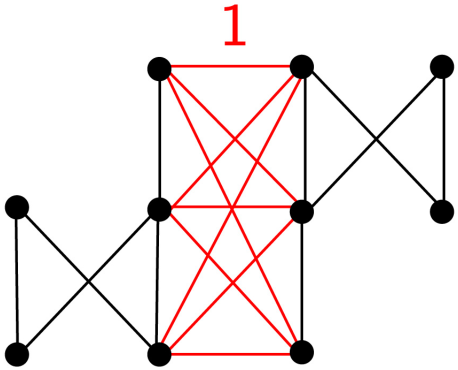

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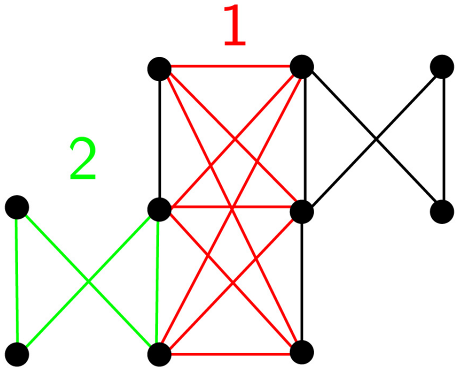

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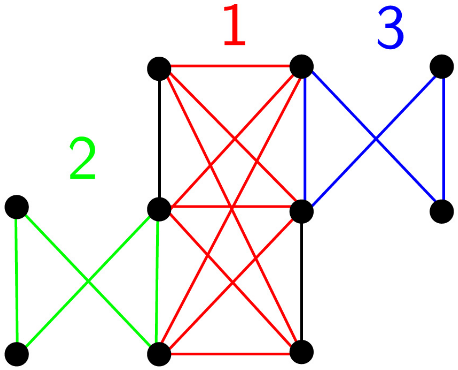

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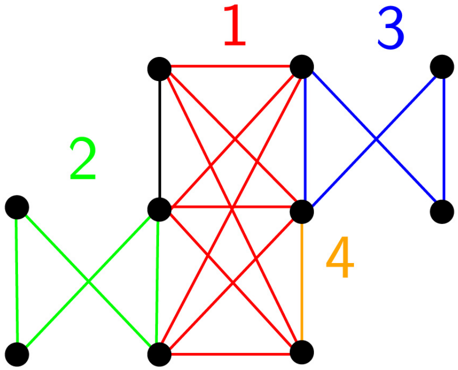

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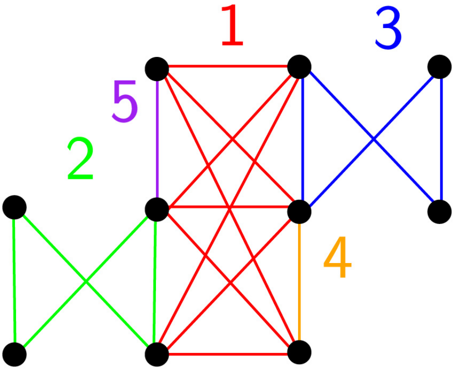

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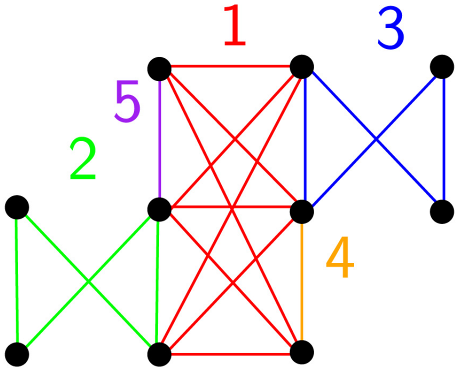

G

$bp(G) = 5$

Graham Pollack

Graham-Pollak theorem, 1971

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$K_n = \biguplus_{i=1}^{k} B_i \Leftrightarrow \mathrm{Adj}(K_n) = \sum_{i=1}^{k} \mathrm{Adj}(B_i)$.

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$$\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{\text{rank}=n} = \underbrace{\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}}_{\text{rank}=2} + \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{rank}=2} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{rank}=2}$$

Alon-Saks-Seymour conjecture

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$\chi \leq \mathrm{bp} + 1.$

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Counter-example (Huang, Sudakov '10)

There exists G such that $\chi \geq \mathrm{bp}^{6/5}$.
Upper bound : $\chi \leq \mathcal{O}(\mathrm{bp}^{\log \mathrm{bp}})$.

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Question : Polynomial Alon-Saks-Seymour conjecture

Does there exists P such that for all G, $\chi \leq P(\mathrm{bp})$.

Equivalence

Theorem (B., Lagoutte, Thomassé)

The following statements are equivalent :

- There is a polynomial P such that for all graphs G, $\chi \leq P(\mathrm{bp})$.
- For every graph G, there is a polynomial CS-separator.

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Remark : One direction was already known.

Hierarchy of bp_i

Definition

bp means that every edge can be covered once.

bp_i means that every edge can be covered at most i times.

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There is a polynomial P such that for all graphs G, $\chi \leq P(\mathrm{bp})$ iff for every i, there is a polynomial P such that for all graphs G, $\chi \leq P(\mathrm{bp}_i)$.

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A particular case : oriented bp

bp^o means that every edge can be covered at most once in each direction.

Remark

$\mathrm{bp}_2 \leq \mathrm{bp}^o \leq \mathrm{bp}$.

A further study of bp^o

$bp(K_n)$

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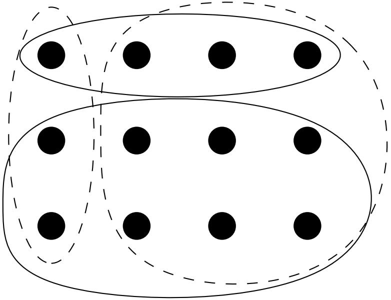

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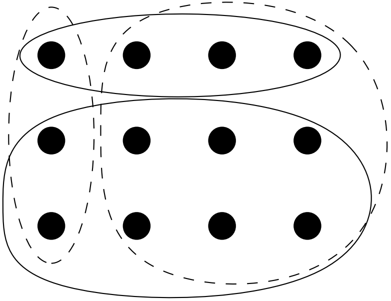

CL-IS and *bp°*

Theorem

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Proof $\Leftarrow$

- Vertices : Pairs (C, S).
- Edges between (C, S) and (C', S') if $x \in C \cap S'$.
- Bipartite packing ? n.

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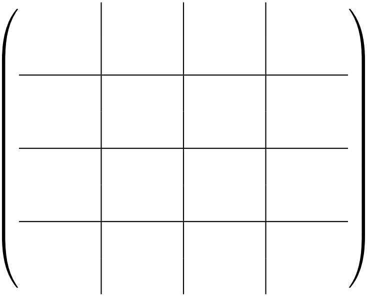
Proof $\Rightarrow$

- Vertices : bipartite graph (A, B).
- Edges : (A, B) and (A', B') if $x \in A \cap A'$.
- There are cuts separating (C_x, S_x).

List-M partition problems

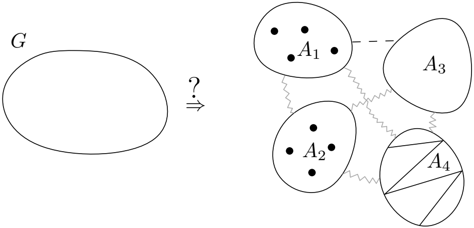


Adj(G)

$\Rightarrow$

$$\begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}$$

Matrix M for the stubborn problem ($k = 4$).



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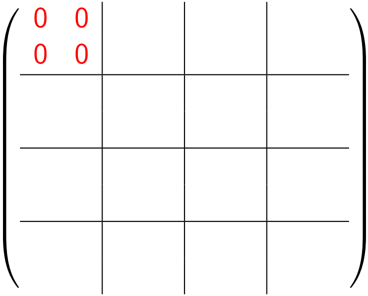

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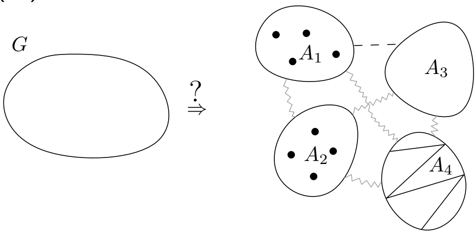

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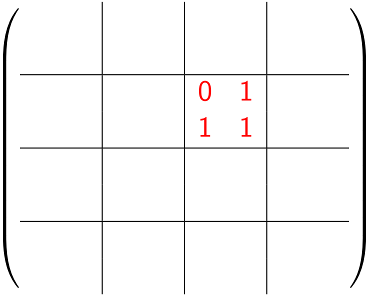

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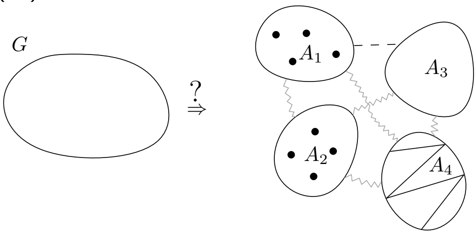

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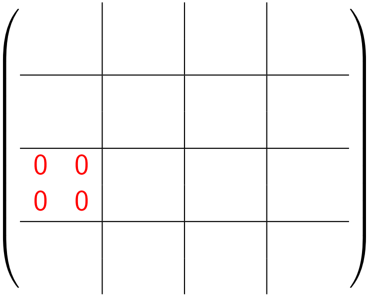

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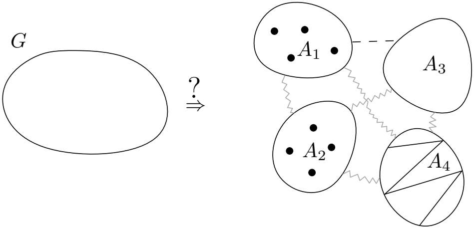

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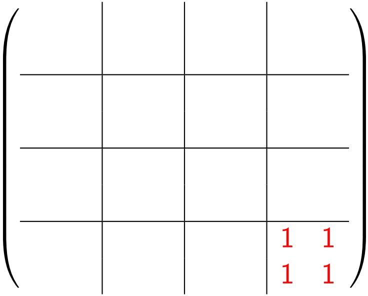

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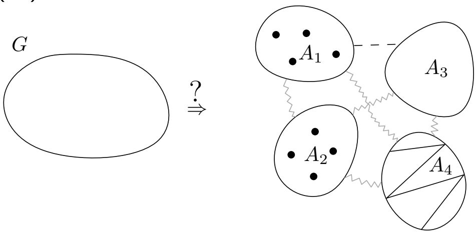

Feder & Hell : Dichotomy theorem ?

Classification P or NP-complete for small matrices, $k \leq 3$.
Classification for $k = 4$ except for the stubborn problem.

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Existing bound

The stubborn problem can be solved in time $\mathcal{O}(n^{\log n})$ via decomposition into $\mathcal{O}(n^{\log n})$ instances of 2-SAT.

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Complexity result

Cygan et al, 2010 : The stubborn problem is in P.

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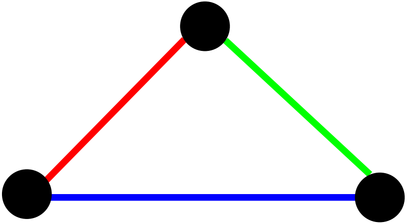
Question

Decomposing the stubborn problem into $P(n)$ instances of 2-SAT ?

3-Compatible Coloring Problem



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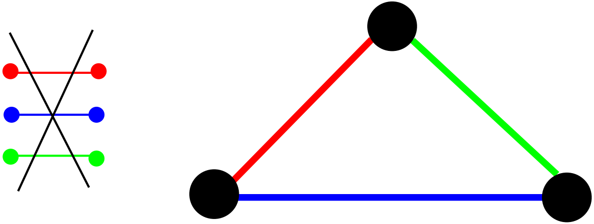

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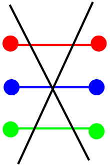

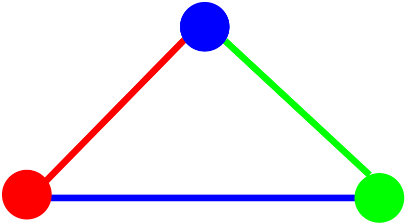

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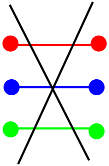

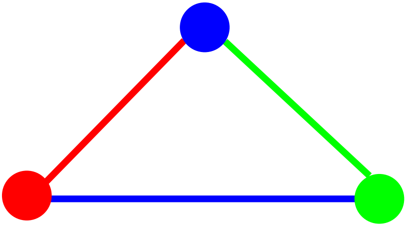

Existing bound

3-CCP can be decomposed into $\mathcal{O}(n^{\log n})$ instances of 2-SAT.

Equivalence theorem

Equivalence theorem

The following are equivalent :

1. There is a polynomial P such that for all graphs G, $\chi \leq P(\mathrm{bp})$.
2. For every integer i, there is a polynomial P such that for all graphs G, $\chi \leq P(\mathrm{bp}_i)$.
3. For every graph G, there is a polynomial CS-separator.
4. For every graph G and every list assignment $\mathcal{L} : V \rightarrow \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on $(G, \mathcal{L})$.
5. For every n and every edge-coloring $f : E(K_n) \rightarrow \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on (K_n, f).

Prospects

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- Study the Clique-Stable separation on perfect graphs thanks to structure theorem.

Questions

Thanks for your attention.