Hamilton cycles in the random geometric graph

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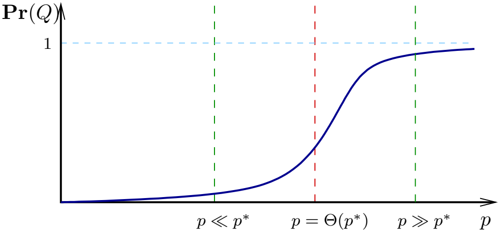
Random graphs and related models

- Erdős-Rényi models: $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$
- Variations: fixed degree sequence, regular graphs...
- Power-law degree sequence: inhomogeneous random graphs, preferential attachment, internet graph...
- Random boolean formulas
- Statistical mechanics: Ising model, Potts model
- Proximity graphs: **Random geometric graphs**, nearest neighbour graphs, Delaunay graphs, models of wireless networks...

Threshold functions

Usual setting:

- $n \to \infty$ vertices,
- $p = p(n)$ “density” parameter,
- property Q



Example: existence of triangles

Threshold functions (sharp)

Usual setting:

- $n \to \infty$ vertices,
- $p = p(n)$ “density” parameter,
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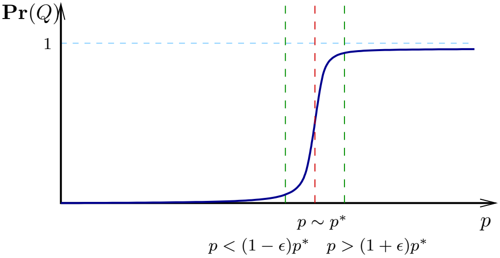


Example: giant component, connectedness, Hamilton cycles...

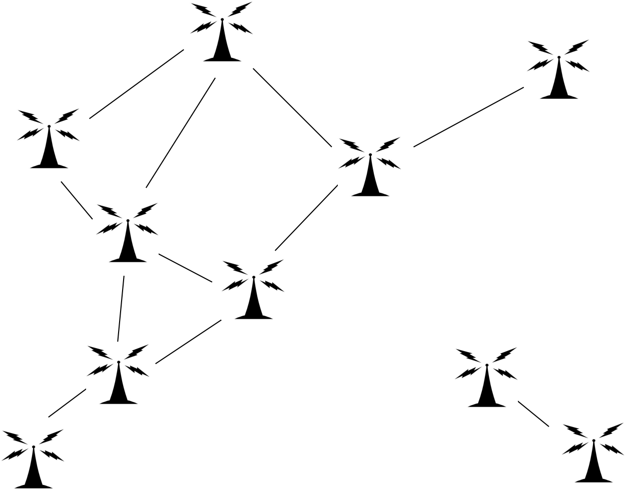
Random graph process

- Start with empty graph on n vertices; add edges one by one; end with the complete graph

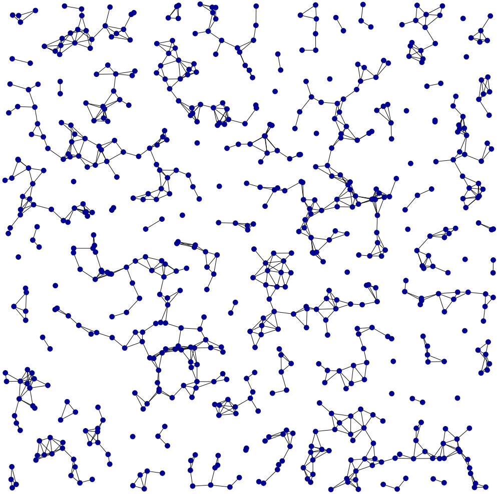
$$\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m, \ldots, \mathcal{G}_{\binom{n}{2}}$$

- $\mathcal{G}_m$ is distributed like $\mathcal{G}(n, m)$
- We look for “hitting time” properties

Wireless networks



Random geometric graph



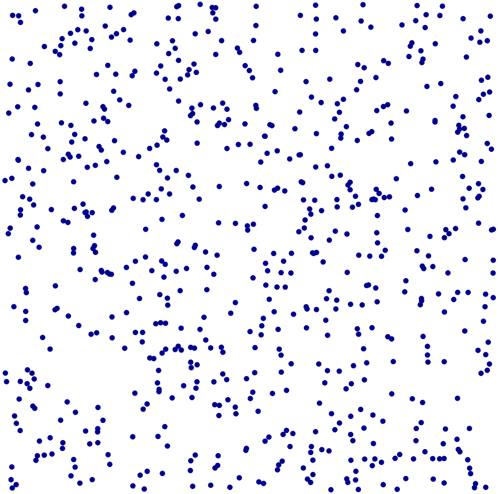
(Gilbert 1961)

n vertices

radius $r = r(n)$

$n \to \infty$

Random process: $0 \leq r \leq \sqrt{2}$



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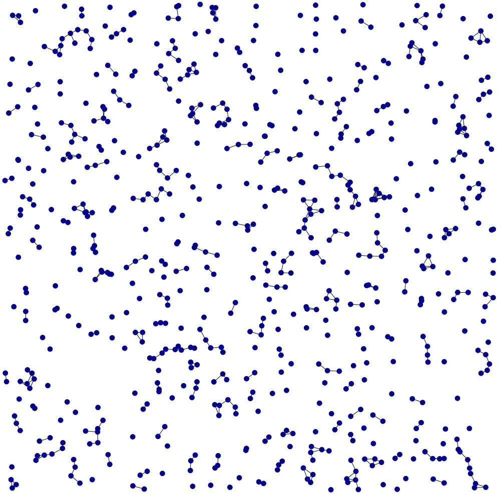

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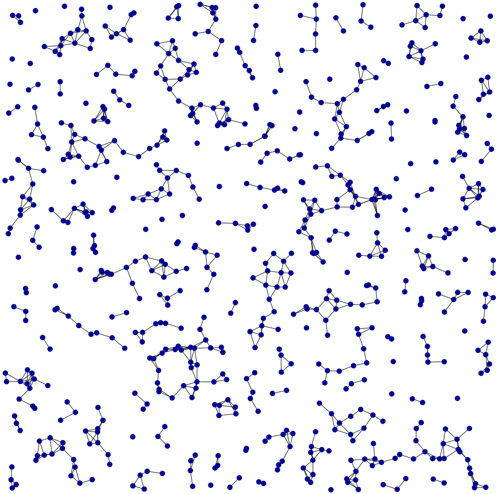

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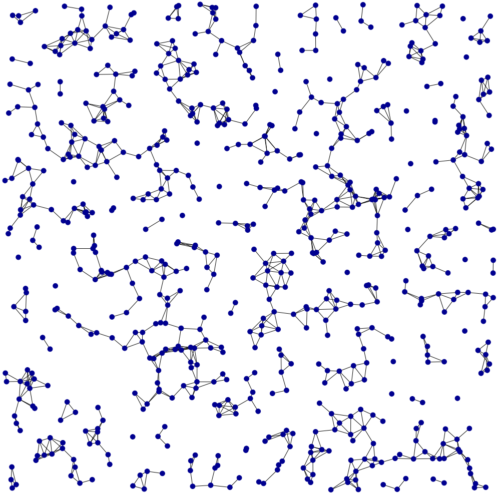
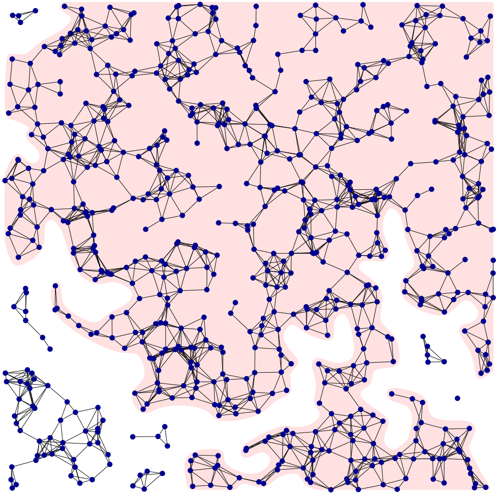
Random process: $0 \le r \le \sqrt{2}$



no giant component yet

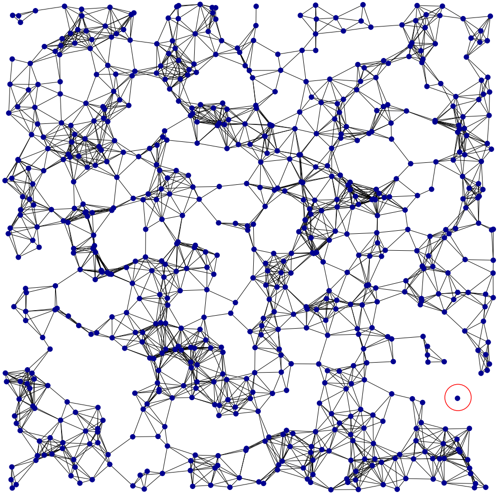
Random process: $0 \le r \le \sqrt{2}$



$r \sim \sqrt{C/n}$

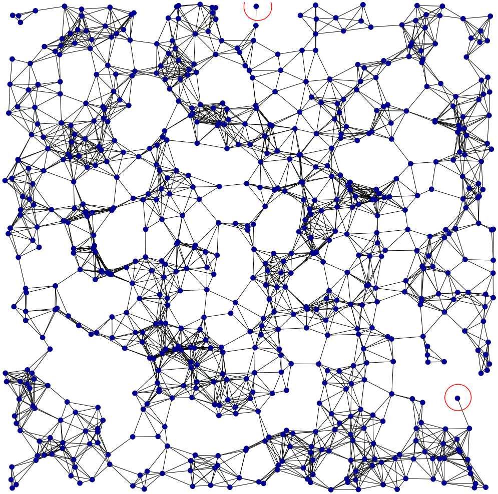
giant component!

Random process: $0 \leq r \leq \sqrt{2}$



still disconnected!

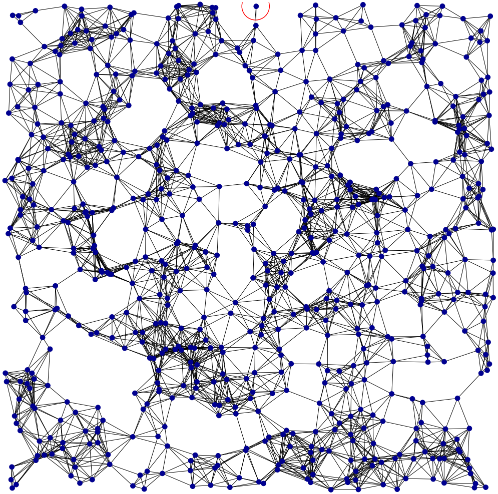
Random process: $0 \leq r \leq \sqrt{2}$



connected
=
no isolated vertices
(a.a.s.)

$$r = \sqrt{\frac{\log n + O(1)}{\pi n}}$$

Random process: $0 \leq r \leq \sqrt{2}$



2-connected

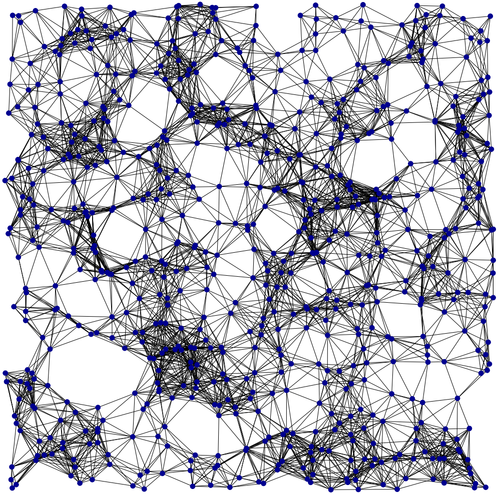
=

no deg. 1 vertices

(a.a.s.)

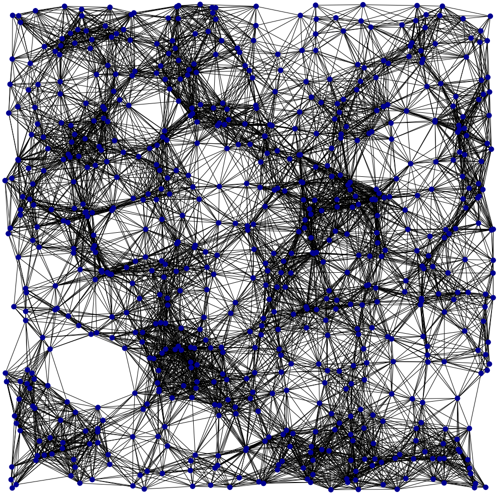
$$r = \sqrt{\frac{\log n + \log\log n + O(1)}{\pi n}}$$

Random process: $0 \leq r \leq \sqrt{2}$



higher connectivity

Random process: $0 \leq r \leq \sqrt{2}$

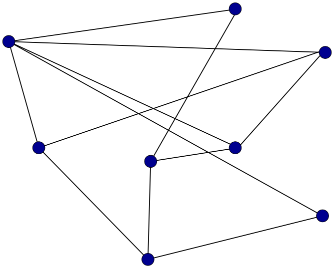


still large diameter:

$\Theta(1/r)$

bad expansion

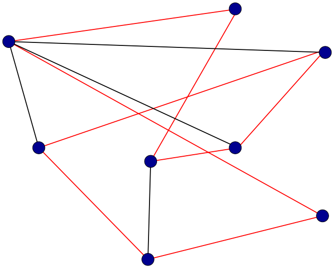
What about hamilton cycles?



Necessary conditions: min. deg. ≥ 2, 2-connectivity

Are they sufficient for the RGG?

What about hamilton cycles?



Necessary conditions: min. deg. ≥ 2, 2-connectivity

Are they sufficient for the RGG?

Earlier results

Thm (Petit 2001)

The RGG with $r = \sqrt{\omega(\log n)/n}$ has a.a.s. a Hamilton cycle.

Thm (Díaz, Mitsche & P.G. 2007)

For any $\epsilon > 0$, the RGG with $r \geq (1+\epsilon)\sqrt{\frac{\log n}{\pi n}}$ has a.a.s. a Hamilton cycle.
(extension to general ℓ_p norm)

Recent results

Thm (Ballogh, Bollobás, Krivelevich, Müller, P.G., Walters & Wormald 2010)

In the RGG process:
Hamiltonian $\iff$ min. deg. ≥ 2 (a.a.s.)
(extension to general dimension and ℓ_p norm)

Thm (Ballogh, Bollobás & Walters 2010)

Weaker analogue for the k-Nearest Neighbour Graph.

Thm (Krivelevich & Müller 2010)

Pancyclic $\iff$ min. deg. ≥ 2 (a.a.s.)

Recent results

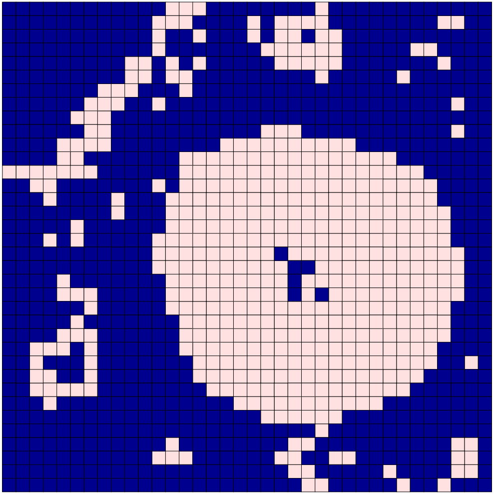
Thm (Ballogh, Bollobás, Krivelevich, Müller, P.G., Walters & Wormald 2010)

In the RGG process:
Hamiltonian $\Longleftrightarrow$ min. deg. ≥ 2 (a.a.s.)
(extension to general dimension and ℓ_p norm)

Thm (Müller, P.G. & Wormald 2010)

$k/2$ disjoint Hamilton cycles $\Longleftrightarrow$ min. deg. $\geq k$ (a.a.s.)
(extension to general dimension and ℓ_p norm)

Main ideas: tesselation



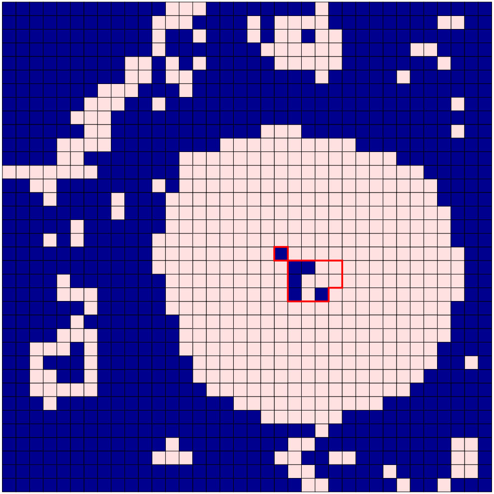
Set $r = \sqrt{\frac{\log n}{\pi n}}$
(not 2-connected)

$\square \updownarrow \delta r$

■ dense ($\geq M$ points)

□ sparse ($< M$ points)

Main ideas: tesselation



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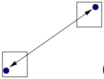
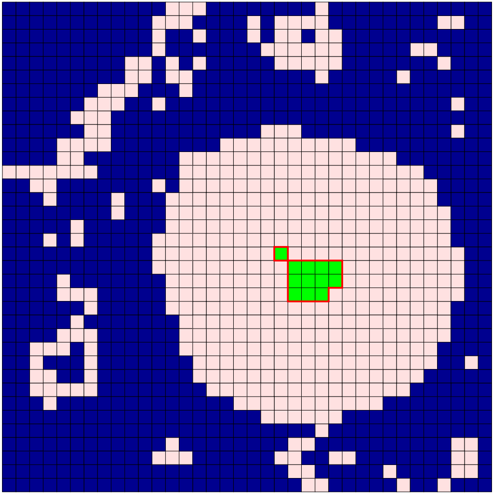


dist. $\leq r$

Main ideas: tesselation



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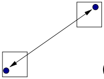

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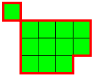
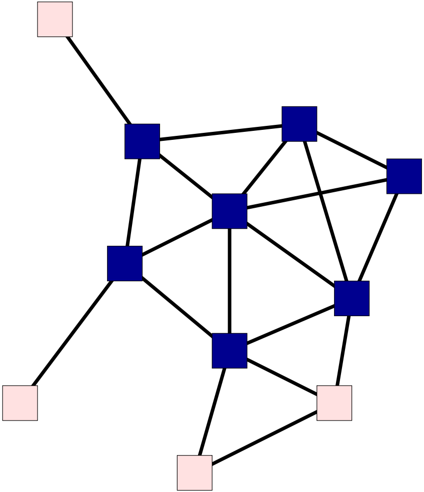


bad cells

Main ideas: large scale template



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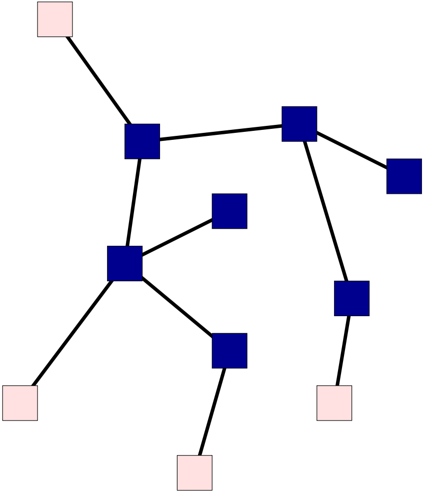

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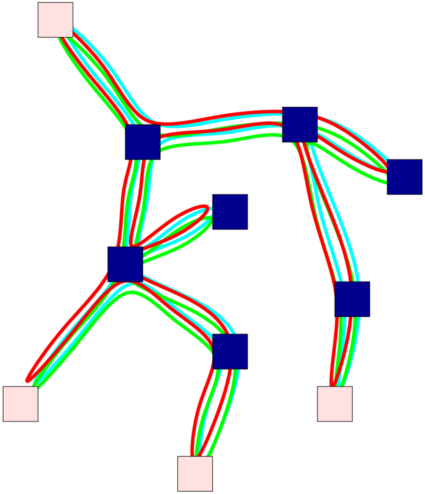
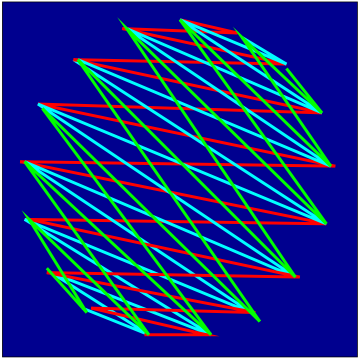


Main ideas: rerouting I



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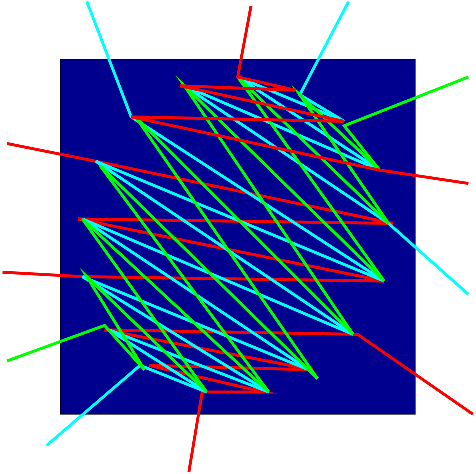
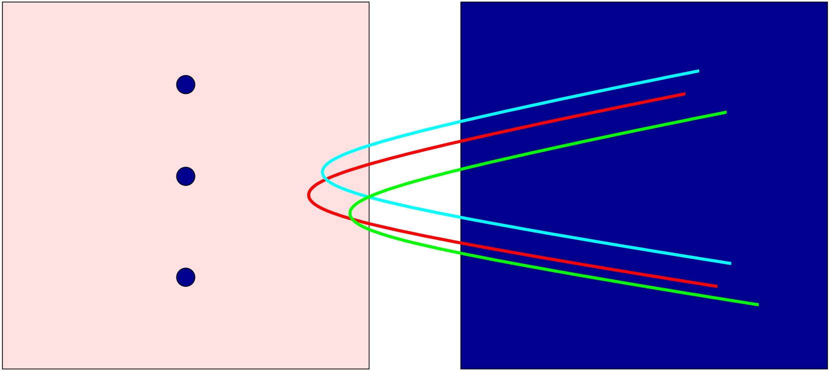


Main ideas: rerouting II



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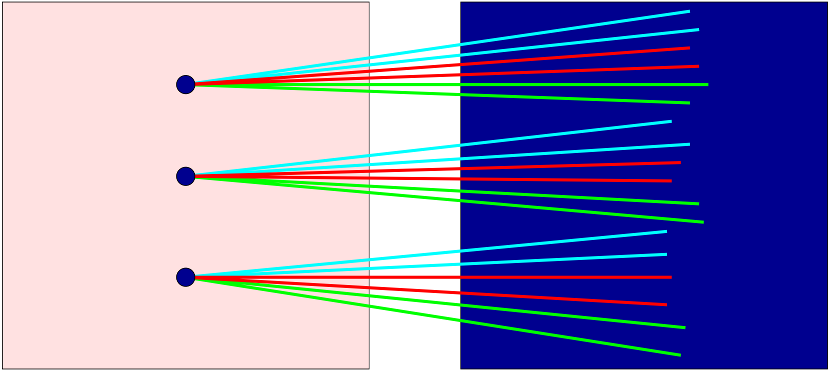

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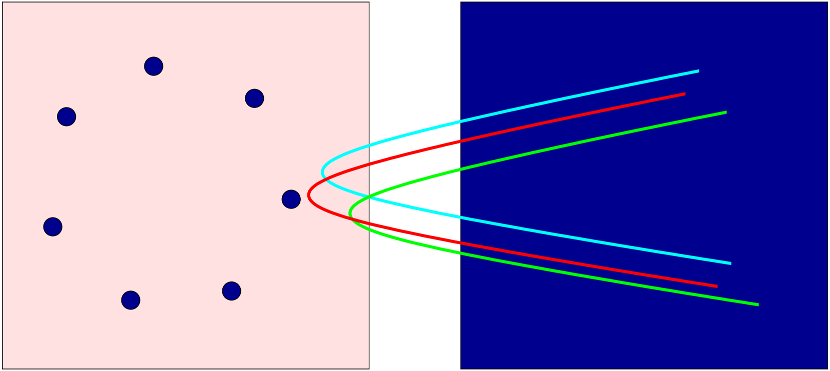

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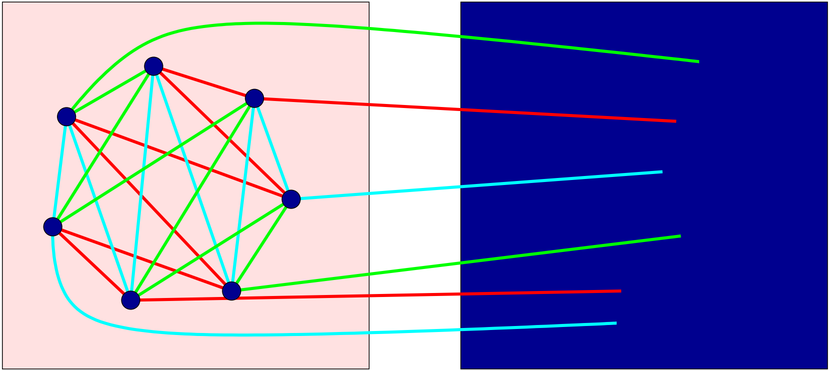
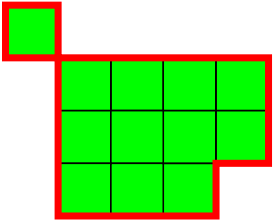


Main ideas: rerouting III



a bit harder!

Open question

Are there always $\lfloor \frac{\delta(RGG)}{2} \rfloor$ edge disjoint Hamilton cycles?

thank you!