A logical approach to matroid decomposition

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Matroids have been design to abstract the notion of dependence.

Definition

A matroid is a pair $(E, \mathcal{I})$, E is a finite set and $\mathcal{I}$ is included in the power set of E. Elements of $\mathcal{I}$ are said to be independent sets, the others are dependent sets.
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1. $\varnothing \in \mathcal{I}$
2. If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$
3. If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

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$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Here the set $\{1, 2, 4\}$ is independent and $\{1, 2, 3\}$ is dependent.

The second example is the cycle matroid of a graph.

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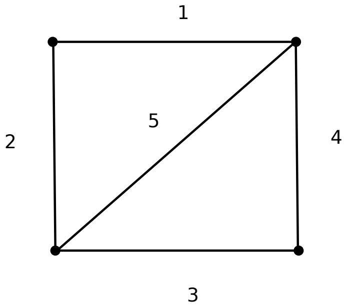

Here the set $\{1, 2, 4\}$ is independent whereas $\{1, 2, 3, 4\}$ and $\{1, 2, 5\}$ are dependent.

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Any cycle matroid is a representable matroid, i.e. it is isomorph to a vector matroid.

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The dependence relation is the same over the edges and over the vectors representing the edges.

This matrix represents the former graph:

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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In a cycle matroid it is a spanning tree of the graph.

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A set $\mathcal{C}$ of subsets of the set E is the collection of circuits of a matroid if and only if :

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2. $C_1, C_2 \in \mathcal{C}^2$ *if* $C_1 \subseteq C_2$ *then* $C_1 = C_2$
3. $C_1, C_2 \in \mathcal{C}^2$, $e \in C_1 \cap C_2 \Rightarrow \exists C \in \mathcal{C}, C \subseteq C_1 \cup C_2 \setminus \{e\}$

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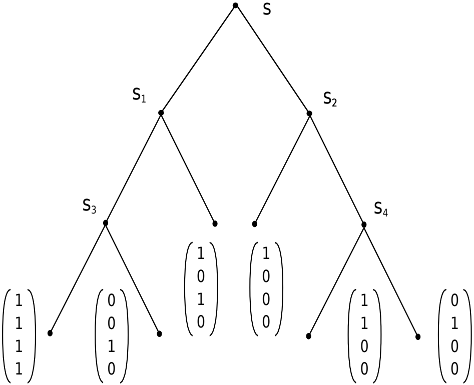
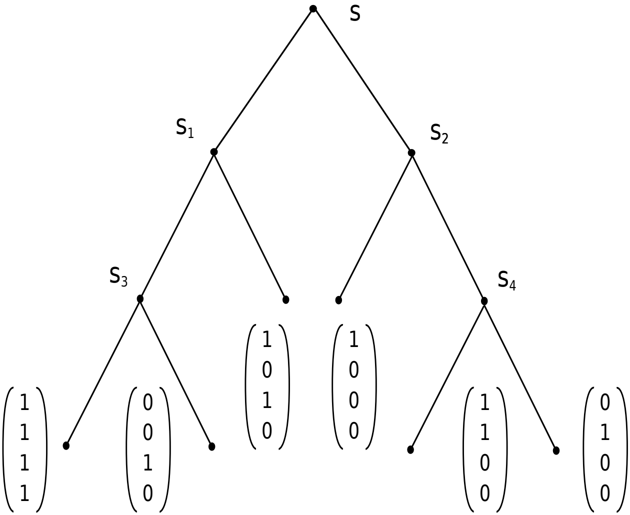


Three important spaces are defined at each node s of the tree :

- E_s is the subspace generated by all the leaves of the tree rooted in s


S
S1
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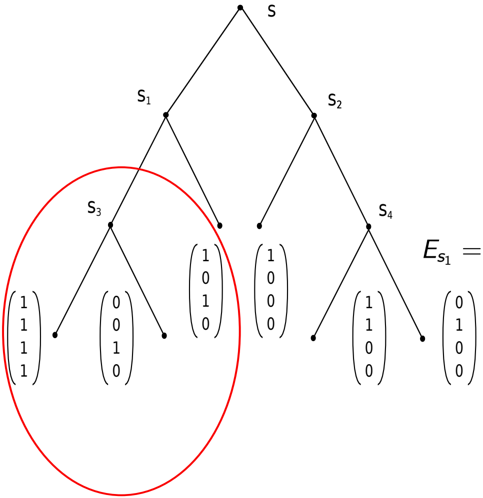

$$E_{s_1} = < \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

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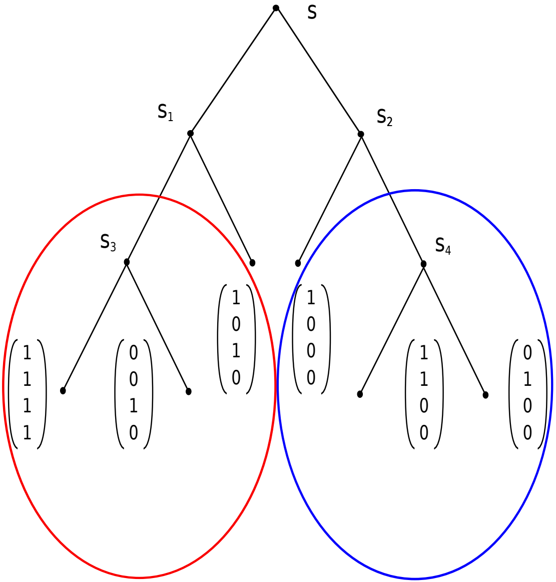
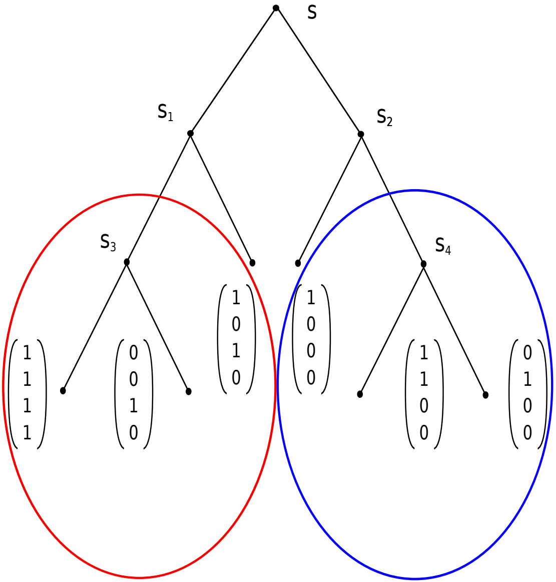
- E_s^* is the subspace generated by all the leaves not in the tree rooted in s



$$E^*_{s_1} = < \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} >$$

Three important spaces are defined at each node s of the tree :

- E_s is the subspace generated by all the leaves of the tree rooted in s
- E_s^* is the subspace generated by all the leaves not in the tree rooted in s
- B_s is the intersection of E_s and E_s^*



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The width at s is the dimension of B_s and the width of the decomposition is the maximum over all nodes.

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Theorem

There is an fpt algorithm which computes a branch decomposition of a representable matroid A of width at most $3t$ if $bw(A) \leq t$. If $bw(A) > t$, the algorithm halts without output.

The following relations define the monadic second order theory on matroids, called MSO_M, which is inspired by the MSO_2 logic over the graphs.

1. $=$, the equality for element and set of the matroid
2. $e \in F$, where e is an element of the set F
3. $indep(F)$, where F is a set and the predicate is true iff F is an independent set of the matroid

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3. $indep(F)$, where F is a set and the predicate is true iff F is an independent set of the matroid

The fact of being a circuit is definable in this logic.

$$Circuit(X) = \neg indep(X) \wedge \forall Y \, (Y \nsubseteq X \vee X = Y \vee indep(Y))$$

We now want to prove the following theorem :

Theorem

The model checking problem for MSO_M is decidable in time $f(t, k, l) \times n^3$ over the set of representable matroids, where f is a computable function, k the size of the field, t the branch-width and l the size of the formula.

We now want to prove the following theorem :

Theorem

Let M be a matroid of branch-width less than t, $\bar{T}$ one of its enhanced tree and $\phi(\vec{x})$ a MSO_M formula with free variables $\vec{x}$, we have

$$(M, \vec{a}) \models \phi(\vec{x}) \Leftrightarrow (\bar{T}, f(\vec{a})) \models F(\phi(\vec{x}))$$

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For each node s of a decomposition tree we compute a base of B_s, and we put in a *characteristic matrix* of s the bases of its boundary subspace and the ones of its two children.

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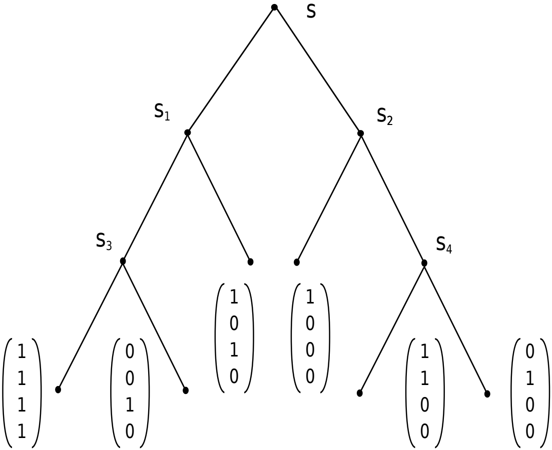
Definition (Enhanced branch decomposition tree)

Let T be a branch decomposition tree of the matroid represented by A, an enhanced branch decomposition tree is T with, on each node, a label representing a characteristic matrix at this node.


S
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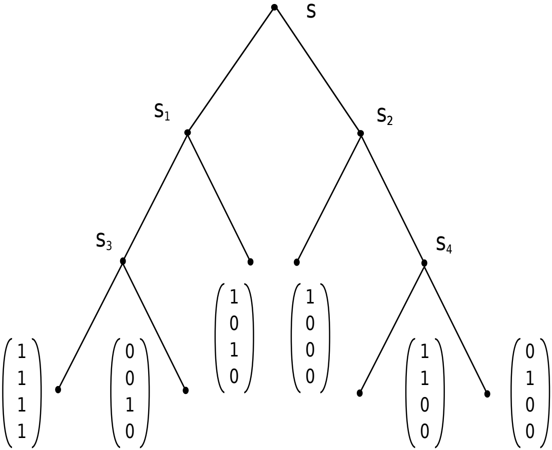
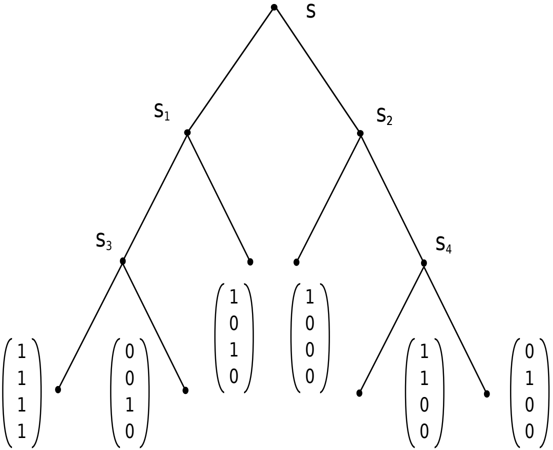
0 1 0 0



$$B_{s_3} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$C_{s_3} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

$$N_{s_3} = \left(\begin{array}{c|c|c} 0 & 1 & 1 \end{array} \right)$$



$$B_{s_4} = < \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} >$$

$$C_{s_4} = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

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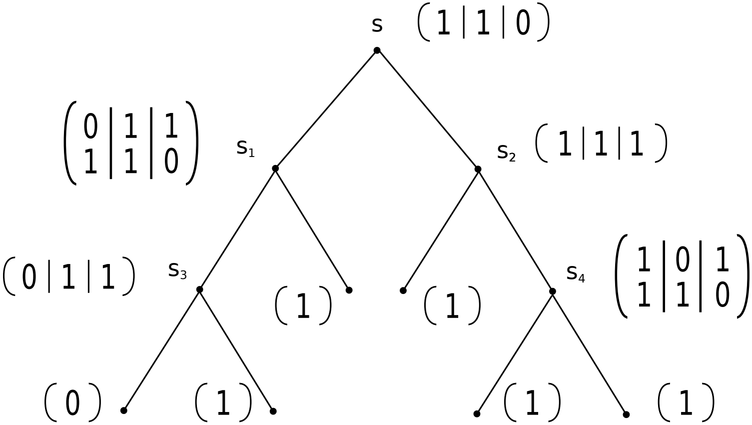
s
(1 | 1 | 0)
s1
s2
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s3
s4
(0 | 1 | 1)
(1 | 0 | 1 ; 1 | 1 | 0)
(1)
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(0)
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Definition (Signature)

A signature is a sequence of elements of $\mathbb{F}$, denoted $\lambda = (\lambda_1, \ldots, \lambda_l)$.

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Definition (Signatures of a set)

Let A be a matrix representing a matroid and T one of its enhanced tree. Let s be a node of T, X a subset of the columns of A in bijection with the leaves of T_s. Let $c_1, \ldots, c_l$ denote the vectors of the third part of C_s, which are a base of B_s. Let v an element of B_s, obtained by a non trivial linear combination of elements of X. If v is written $\sum_i \lambda_i c_i$ in the base $c_1, \ldots, c_l$, we say that X admits the *signature* $\lambda = (\lambda_1, \ldots, \lambda_l)$ at s. X also always admits $\varnothing$ as signature at s.

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad C_{s_1} = \left(\begin{array}{c|c|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

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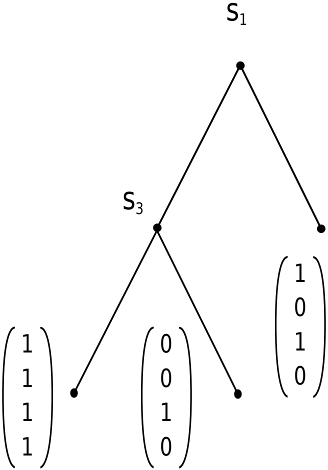
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad X \text{ admits } (1)$$

Lemma

Let T be an enhanced tree, s one of its nodes with children s_1, s_2 and $N_s = (N_1|N_2|N_3)$ the label of s. X_1 and X_2 are respectively elements in bijection with leaves of T_{s_1} and T_{s_2}. Assume we have the relation

$$\sum \mu_i N_1^i + \sum \gamma_j N_2^j = \sum \lambda_k N_3^k \qquad (1)$$

then $X = X_1 \cup X_2$ admits λ at s if and only if X_1 admits μ at s_1 and X_2 admits γ at s_2.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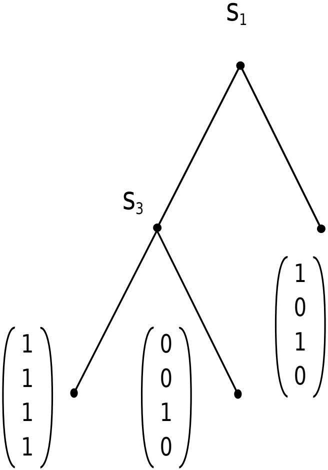

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Theorem (Characterization of dependency)

Let A be a matrix representing a matroid, T one of its enhanced tree and X a set of column of A. X is dependent if and only if there exist a signature λ_s for each node s of the tree T such that :

1. *if s_1 and s_2 are the children of s labeled by N then λ_s, λ_{s_1}, λ_{s_2} and N satisfy Equation 1*
2. *the set of leaves of signature non $\emptyset$ is a non empty subset of X*
3. *the signature at the root is $(0, \ldots, 0)$*

- A signature is represented by $\vec{X}_\lambda$ indexed by all the signatures of size less than t.

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- The number of such variables is bounded by a function in t.
- Consistency :

$$\Omega(\vec{X_\lambda}) = \forall s \bigvee_{\lambda} \left(X_\lambda(s) \bigwedge_{\lambda' \neq \lambda} \neg X_{\lambda'}(s) \right)$$

The signature satisfy Equation 1 represented by the predicate θ:

$$\Psi_1(s, \vec{X_\lambda}) = \exists s_1, s_2\; lchild(s, s_1) \wedge rchild(s, s_2)$$

$$\bigwedge_{\lambda_1, \lambda_2, \lambda, N} (label(s) = N \wedge X_{\lambda_1}(s_1) \wedge X_{\lambda_2}(s_2) \wedge X_\lambda(s)) \Rightarrow \theta(N, \lambda_1, \lambda_2, \lambda)$$

The set of leaves of signature non $\varnothing$ is a non empty subset of X :

$$\Psi_2(X, \vec{X_\lambda}) = \forall s[(leaf(s) \wedge \neg X_\varnothing(s)) \Rightarrow (X(s) \wedge X_{(1)}(s))]$$

$$\wedge \exists u\,(leaf(u) \wedge \neg X_\varnothing(u))$$

The signature at the root is $(0, \ldots, 0)$:

$$\Psi_3(\vec{X_\lambda}) = \exists s\, root(s) \wedge X_{(0,\ldots,0)}(s)$$

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By induction we translate ϕ a MSO_M formula over a matroid into $F(\phi)$ a MSO formula over enhanced tree.

We have then proved

Theorem

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$$(M, \vec{a}) \models \phi(\vec{x}) \Leftrightarrow (\bar{T}, f(\vec{a})) \models F(\phi(\vec{x}))$$

Definition (Spectrum)

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A set X of integers is said to be *ultimately periodic* if there are two integers a and b such that, for $n > a$ in X we have $n = a + k \times b$.

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Theorem

Let ϕ a formula of MSO_M, then the spectrum of ϕ restricted to matroids of branch-width t is ultimately periodic.

Theorem (Courcelle)

Let $\phi(X_1, \ldots, X_n)$ be a MSO formula with free variables. For every tree t, there exists a linear delay enumeration algorithm of the $X_1, \ldots, X_n$ such that $t \models \phi(X_1, \ldots, X_n)$ with preprocessing time $\mathcal{O}(|t| \times ht(t))$.

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Corollary

Let $\phi(X_1, \ldots, X_n)$ be an MSO_M formula, for every matroid of branch-width t, the enumeration of the sets satisfying ϕ can be done with linear delay after a cubic preprocessing time.

All the previous theorems also work with colored matroids and colored tree.

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Generalisation of very natural problems and decidable in linear time over matroids of bounded branch-width.

How to glue matroids together to form new matroids ?

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Definition (Boundaried matroid)

A pair (M, γ) is called a t boundaried matroid if M is a matroid and γ is an injective function from $[|1, t|]$ to M whose image is an independent set. The elements of the image of γ are called boundary elements and the others are called internal elements.

Example of an operation on representable matroid

$N_1 = (M_1, \gamma_1)$ and $N_2 = (M_2, \gamma_2)$ are two t boundaried representable matroids represented by the set of vectors A_i in the vector space E_i. $E_1 \times E_2$ is the direct product of the two vector spaces and $\langle \{\gamma_1(j) - \gamma_2(j)\} \rangle$ is the subspace generated by the elements of the form $\gamma_1(j) - \gamma_2(j)$.

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Definition

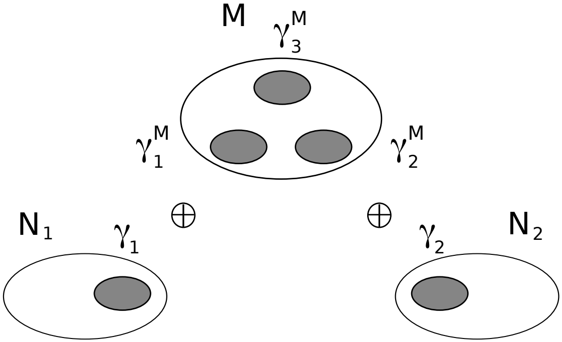
Let E be the quotient space of $(E_1 \times E_2)$ by $\langle \{\gamma_1(j) - \gamma_2(j)\} \rangle$. There are natural injections from A_1 and A_2 into $E_1 \times E_2$ and then in E. The elements of $A = (A_1, \gamma_1) \overline{\oplus} (A_2, \gamma_2)$ are the images of A_1 and A_2 by these injections minus the boundary elements. The dependence relation is the linear dependence in E.

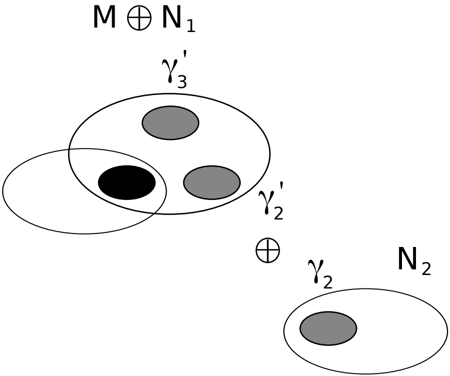
A matroid M which is partitioned in three independent sets $\gamma_i'([|1,t_i|])$ with $t_i \leq t$ for $i = 1, 2, 3$ is called a *3-partitioned matroid*.

A matroid M which is partitioned in three independent sets $\gamma'_i([|1, t_i|])$ with $t_i \leq t$ for $i = 1, 2, 3$ is called a 3-*partitioned matroid*.

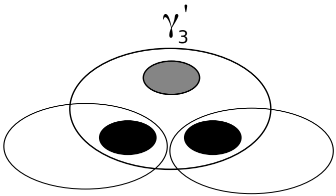
Definition

Let $\overline{N_1} = (N_1, \gamma_1)$ and $\overline{N_2} = (N_2, \gamma_2)$ be respectively a t_1 and a t_2 boundaried matroids. $\overline{N} = \overline{N_1} \odot_M \overline{N_2}$ is a t_3 boundaried matroid defined by :
$(\overline{N_1} \oplus (M, \gamma'_1), \gamma'_2) \oplus \overline{N_2}$ with boundary γ'_3.


M
γ3M
γ1M
γ2M
⊕
⊕
N1
γ1
γ2
N2


M ⊕ N1
γ'3
γ'2
⊕
γ2
N2

$$N_1 \odot_M N_2$$



Definition (Terms)

Let $\mathcal{L}$ a finite set of boundaried matroids and $\mathcal{M}$ a finite set of 3-partitioned matroids. A term of $T(\mathcal{L}, \mathcal{M})$ and its value are recursively defined in the following way :

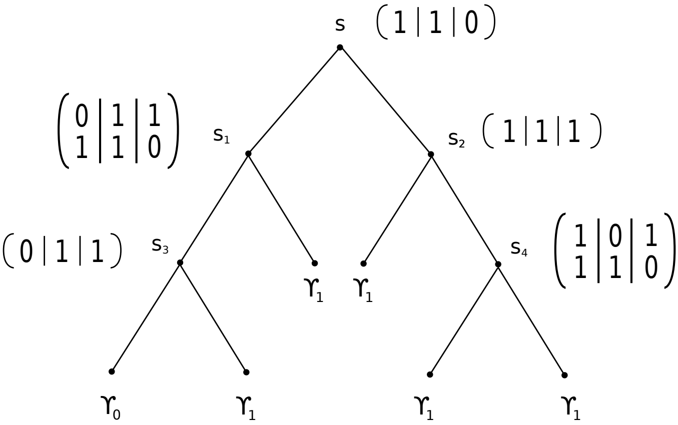
- ϵ is a 0 term whose value is the empty matroid
- an element of $\mathcal{L}$ with a boundary of size t is a term whose value is itself
- Let T_1 and T_2 be two terms of value M_1 and M_2 which are a t_1 and a t_2 boundaried matroids and $M \in \mathcal{M}$ partitioned in three sets of cardinality t_1, t_2 and t_3. MT_1T_2 is a term whose value is the t_3 boundaried matroid $M_1 \odot_M M_2$.

$T(\Upsilon, \mathcal{M}_t^{\mathbb{F}})$:

$\mathcal{M}_t^{\mathbb{F}}$ is the set of $3t$ partitioned matrix on the field $\mathbb{F}$

Υ the set containing the two following matrices with a boundary :

- $\varUpsilon_0$ is the matrix $\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right)$.
- $\varUpsilon_1$ is the matrix $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$.


s
(1 | 1 | 0)
s1
(0 | 1 | 1 ; 1 | 1 | 0)
s2
(1 | 1 | 1)
s3
(0 | 1 | 1)
s4
(1 | 0 | 1 ; 1 | 1 | 0)
ϒ1
ϒ1
ϒ0
ϒ1
ϒ1
ϒ1

We have built something we already know :

Theorem

A finitely representable matroid is in $T(\Upsilon, \mathcal{M}_t^{\mathbb{F}})$ if and only if it is of branch-width less than t.

Need another operation to have something different.

$$\begin{array}{ccc} B & \xrightarrow{\gamma_1} & M_1 \\ {\scriptstyle \gamma_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ M_2 & \xrightarrow[i_2]{} & M_1 \oplus M_2 \end{array}$$

The set B is an independent set of size t. $i_1 \circ \gamma_1 = i_2 \circ \gamma_2$ where γ_1 and γ_2 are injective their images are the boundaries and therefore independent sets of M_1 and M_2.

Figure: The diagram of the pushout

A set $\mathcal{D}$ is the set of dependent sets of a matroid if it satisfies :

- (A_1) : $D_1, D_2 \in \mathcal{D}^2$, $e \in D_1 \cap D_2 \Rightarrow D_1 \cup D_2 \setminus \{e\} \in \mathcal{D}$, where $\mathcal{D}$ is the set of its dependent sets.
- (A_2) : $D \in \mathcal{D}, D \subset D' \Rightarrow D' \in \mathcal{D}$

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- $\mathcal{D}$ is the set $\{D_1 \cup D_2 | D_1 \text{ dependent in } M_1 \text{ or } D_2 \text{ dependent in } M_2\}$
- take the closure of $\mathcal{D}$ by the axiom (A_1)

Definition

Let M_1 and M_2 two t boundaried matroids with ground sets S_i and boundaries γ_i. We introduce the elements $\{e_1, \ldots, e_t\}$, which are disjoint from S_1 and S_2. Let $E = S_1 \cup S_2 \cup \{e_1, \ldots, e_t\} \setminus \{\gamma_1([|1,t|]) \cup \gamma_2([|1,t|])\}$. Let $\mathcal{D}$ be the set $\{D_1 \cup D_2 | D_1 \text{ dependent in } M_1 \text{ or } D_2 \text{ dependent in } M_2\}$ where $\gamma_1(i)$ and $\gamma_2(i)$ are changed in e_i. Then $M_1 \oplus M_2 = (E, \overline{\mathcal{D}})$.

From this operation we define $\odot_M$ operators and terms.

From this operation we define $\odot_M$ operators and terms.

Let $\mathcal{L}_k$ be the abstract matroids of size less or equal to k.

Let $\mathcal{M}_t$ be 3 partitioned matroids of size less than $3t$.

We study the terms of $T(\mathcal{L}_k, \mathcal{M}_t)$.

The signature returns :

Definition (Signature)

Let T be a term of value M a boundaried matroid and X a set of elements of M. The signature λ_s of the set X at s a node of T is the set of all the subsets A of the boundary such that $X \cup A$ is a dependent set in the matroid vale of T_s. The elements of the boundary are represented in the signature by their index.

Theorem (Characterization of dependency)

Let T be a term of $T(\mathcal{L}_k, \mathcal{M}_t)$ representing the matroid M and X a set of elements of M. X is dependent if and only if there exist a signature λ_s for each node s of T such that :

1. *if s_1 and s_2 are the children of s of label $\odot_N$ then $R(\lambda_{s_1}, \lambda_{s_2}, \lambda_s, N)$*
2. *if s is labeled by an abstract matroid N, then the intersection of X with the elements of N is a set of signature λ_s*
3. *the set of nodes labeled by an abstract matroid of signature non $\varnothing$ is non empty*
4. *the signature at the root contains the empty set*

What is important for the theorem is that the relation R does not depend on X and T.

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The proof relies on the study of the structure of the dependent sets of $M_1 \oplus M_2$.

Theorem

Let M be a a matroid given by $T \in T(\mathcal{L}_k, \mathcal{M}_t)$ and $\phi(\vec{x})$ an MSO_M formula, then $M \models \phi(\vec{a}) \Leftrightarrow T \models F(\phi(f(\vec{a})))$.

Theorem

Let M be a a matroid given by $T \in T(\mathcal{L}_k, \mathcal{M}_t)$ and $\phi(\vec{x})$ an MSO_M formula, then $M \models \phi(\vec{a}) \Leftrightarrow T \models F(\phi(f(\vec{a})))$.

Same proof and translation of $\phi(\vec{x})$ as for the matroids of bounded branch-width !

Thanks for listening!