Maximal Acyclic Subtournaments

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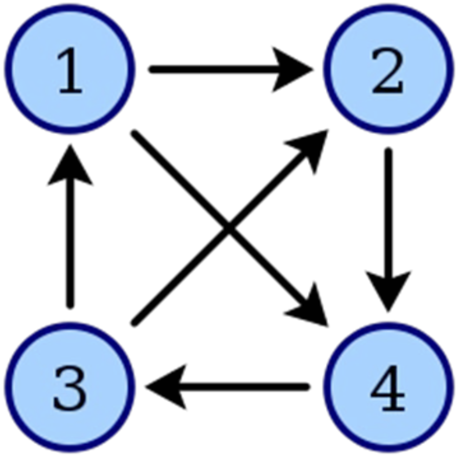
Tournaments & Acyclicity

- **Tournament**: orientation of a complete graph.



- A tournament is **acyclic / transitive** if it does not contain any directed cycles.
- **Maximal** acyclic subtournament: not properly contained in any other acyclic subtournament.

Problem and Results

- How many maximal acyclic subtournaments can a tournament with *n* vertices have?

Application:
Banks winner in Elections

- **Upper Bound**

  Moon (1971): at most 1.717^n.

  Gaspers & M. (2009): at most 1.667^n.

- **Lower Bound**

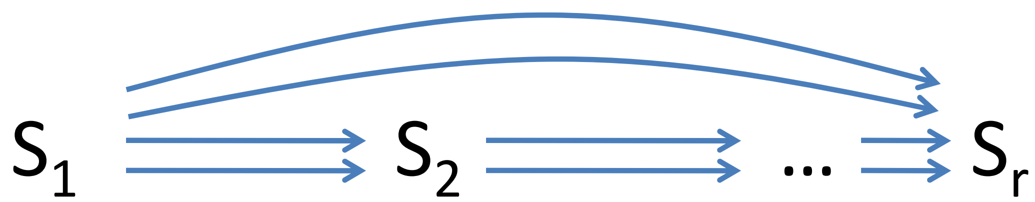
  Moon (1971): at least 1.4570^n.

  Gaspers & M. (2009): at least 1.5448^n.

Strong Tournaments & Acyclicity

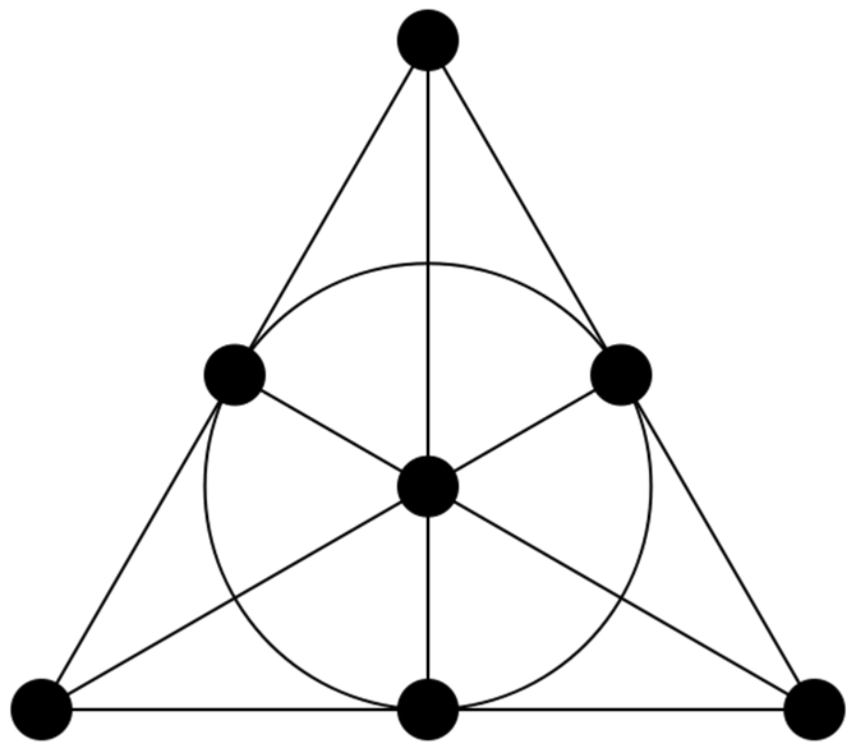
Folklore: tournament acyclic $\leftrightarrow$ no 3-cycles.

A tournament T is **strong** if there is a directed path between any pair of vertices.



$$T = S_1 + S_2 + \ldots + S_r \rightarrow f(T) = \prod f(S_i)$$

Lower bound: $21^{n/7} \approx 1.5448^n$

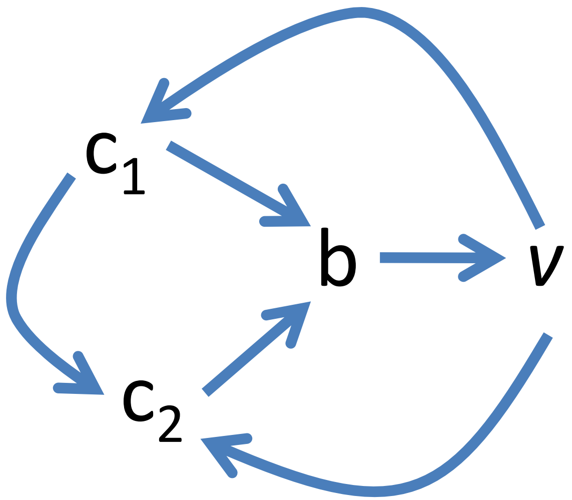


Payley digraph of quadratic residues mod 7
Fano plane generator

Upper Bound I: Branching

Property: Every vertex beats ≤ *n-2* other vertices.

Example Case: *v* beats *n-2* other vertices.



b ∉ W: source *v*, f(n-2) many

b ∈ W, *v* ∈ W: source (b,v), f(n-4) many

b ∈ W, *v* ∉ W:

source c_1 or c_2, 2f(n-3) many

$f(T) \leq f(n-2) + 2f(n-3) + f(n-4) \leq f(n)$ for $f(n) = 1.6181^n$

Upper Bound II: Scores

- The **score** of a vertex *v* is its outdegree.
- The **score sequence** of a tournament is the non-decreasing sequence of out-degrees: $s = s(T) = (s_1,\ldots,s_n)$ with $s_1 \leq \ldots \leq s_n$
- Landau (1953, "On dominance and the structure of animal societies"): *s* is the score sequence of a tournament $\leftrightarrow$

$$\sum_{v=1}^{k} s_v \geq \binom{k}{2} + 1 \quad \text{for all} \quad k = 1, \ldots, n-1, \text{ and}$$

$$\sum_{v=1}^{n} s_v = \binom{n}{2}$$

Upper Bound II: Special Sequences

$S_n = \{s = (s_1,\ldots,s_n): 3 \le s_1 \le \ldots \le s_n \le n-4\}.$

$G: S_n \rightarrow \mathrm{IR}_+,\ G(s) = \sum \beta^{si}$

$\sigma = (3,3,3,3,3,5,7,7,7,7,7)$ if $n = 11$

$\sigma = (3,3,3,3,3,3,8,8,8,8,8,8)$ if $n = 12$

$\sigma = (3,3,3,3,3,3,6,9,9,9,9,9,9)$ if $n = 13$

$\sigma = (3,3,3,3,3,3,4,7,8,\ldots,$
$n-9,n-8,n-5,n-4,n-4,n-4,n-4,n-4,n-4)$ if $n \ge 14$.

Lemma: $G(s) \le G(\sigma)$ for all $s \in S_n$.

Upper Bound II: Convexity

Lemma: $G(s) \leq G(\sigma)$ for all $s \in S_n$.

Technical proof by strict convexity of G.

$\sigma = (3,3,3,3,3,3,6,9,9,9,9,9,9)$ if $n = 13$

Corollary:

$$f(T) \leq G(s) \leq G(\sigma) = 6\beta^3 + \beta^6 + 6\beta^9 \leq \beta^n$$

for $\beta \geq 1.6259$

Lemma: $G(s) \leq G(\sigma)$ for all $s \in S_n$.

$\sigma = (3,3,3,3,3,3,4,7,8,\ldots,n-9,n-8,n-5,n-4,n-4,n-4,n-4,n-4,n-4)$

Prove claims based on G being strictly convex:

Let $s \in S_n$ be a maximizer of G. Then…

Claim 1: score c appears multiple times → $c \in \{3,n-4\}$.

Claim 2: scores $c \in \{3,n-4\}$ appear 2-6 times each.

Claim 3: scores $c \in \{3,n-4\}$ appear exactly 6 times.

Claim 4: $s = \sigma$.

Summary

f(n): maximum number of maximal acyclic subtournaments of a tournament with *n* vertices

- Moon (1971): $1.4570^n \le f(n) \le 1.717^n$
- Gaspers & M. (2009): $1.5448^n \le f(n) \le 1.667^n$

- Exact bounds for small tournaments.

Open Problems (and Conjectures)

- We can improve the bound slightly at the expense of a much longer proof. New technique for better upper bounds?
- Conjecture: $f(n) = 1.5448^n$.
- More general digraphs?
- Approach applicable to Tournament Dominating Sets?