Fixed-Parameter Algorithm for 2-CNF Deletion Problem

Igor Razgon

Computer Science Department

University College Cork

Ireland

A brief introduction to the area of fixed-parameter algorithms

Fixed-parameter algorithms as a way of coping with NP-hardness

Fixed-parameter algorithms allow to solve hard optimization problems:

- exactly,
- in a low-polynomial time.

Of course, they are exponential in the worst case, but:

- the degree of the exponent does not depend on the size of input but on an additional parameter *k* associated with the problem

- in real-world instances the value of the parameter is frequently very small

Definition of a fixed-parameter algorithm

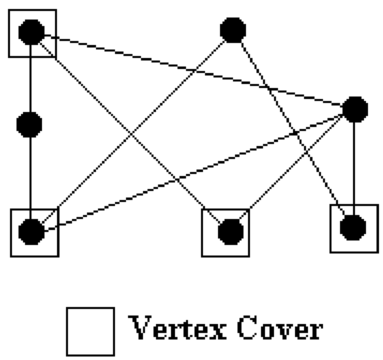
Given an intractable problem with the input size n and a parameter k.

A fixed-parameter algorithm is an algorithm that solves this problem in time $O(f(k)*n^c)$,
where $f(k)$ is an exponential function of k,
c is a constant independent on k.

Example: a fixed-parameter algorithm for the Vertex Cover Problem

Vertex Cover problem:

given a graph *G*, find the smallest Vertex Cover (VC), i.e. a set of vertices incident to all the edges of *G*.



Being parameterized by the size of VC, the problem asks: given a non-negative integer *k*, find out whether *G* has a VC of size at most *k*.

Example: a parameterized algorithm for the Vertex Cover Problem (cont.)

FindVC(G,k)

If *G* has no edges **then** return 'YES'

If *k=0* **then** return 'NO'

Select an edge *{u,v}* of *G*

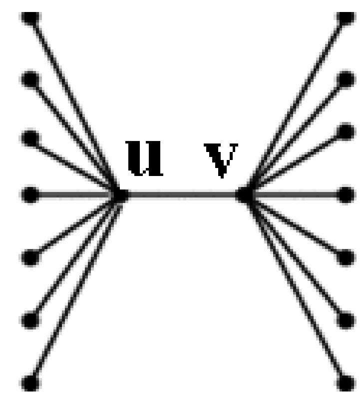
If *FindVC(G\u,k-1)* returns 'YES' **or**

FindVC(G\v,k-1) returns 'YES' **then**

Return 'YES'

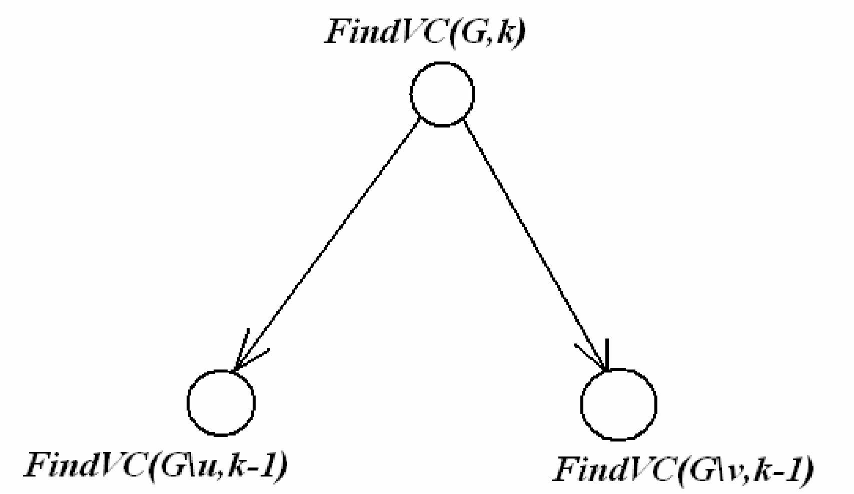
Else

Return 'NO'



Example: a fixed-parameter algorithm for the Vertex Cover Problem (further cont.)

- The recursive applications of *FindVC* can be organized into a search tree.



- The height of the tree is at most k. Each non-leaf node has 2 children. Hence the search tree has $O(2^k)$ nodes. The complexity of *FindVC* is $O(2^k n)$.

- A better algorithm for the VC problem takes $O(1.3^k+n)$. It works in a reasonable time for $k=60$ and a huge n.

- It is much better then to explore all subsets of k vertices.

Fixed-parameter tractability

- A problem that can be solved by a fixed-parameter algorithm is called <u>fixed-parameter tractable</u> (FPT).

- As we have seen, the VC problem parameterized by the size of the output is FPT.

- Another example: many graph-theoretical problems (e.g. Clique) are FPT being parameterized by the treewidth of the underlying graph.

Applications

Fixed-parameter algorithms can be applied in the areas where the problems are associated with parameters that are very small in practice.

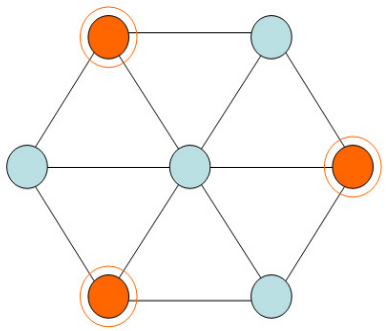
Such areas include:

- Bioinformatics
- Networks Design
- Computer Security
- Machine Learning
- Artificial Intelligence

Fixed-parameter tractability vs. intractability

Parameterized Independent Set Problem

Given a graph *G* and a parameter *k*, find out whether *G* has an independent set (a set of mutually non-adjacent vertices) of size at least *k*.



A simple method of solving the problem.

Select a vertex *v.* Select into the independent set either *v* or one of neighbours of *v* and apply the algorithm recursively to the corresponding residual graph with decreasing the parameter by *1*.

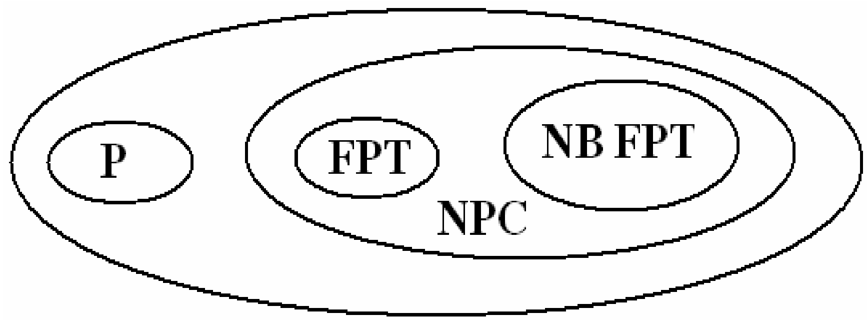
Runtime analysis

The algorithm creates a search tree of height *k*, but the number of children of a node is the number of neighbours of the corresponding vertex plus one. For dense graphs the number of nodes of the tree may be as large as $O(n^k)$. Thus this algorithm is not a fixed-parameter one. This suggests that *there might be no fixed-parameter algorithm solving the problem.*

Fixed-parameter tractability vs. intractability (cont.)

A stronger evidence that the Independent Set Problem is not FPT:

If it is FPT then the widely believed Exponential Time Hypothesis fails (i.e., 3-SAT, Independent Set, and many other problems can be solved in a subexponential time).



NB FPT - not believed to be in FPT

The question of <u>classification</u>:

given an intractable problem, find out whether this problem is FPT or probably not.

The 2-CNF deletion problem

Terminology

An example of a 2-CNF formula: $(X_1 \vee X_2)\ (\neg X_2 \vee \neg X_3)\ (X_3 \vee \neg X_1)$.

- The *clauses* of the formula are $(X_1 \vee X_2)$, $(\neg X_2 \vee \neg X_3)$, and $(X_3 \vee \neg X_1)$.
- The *variables* of the formula are: X_1, X_2, and X_3.
- The *literals* of the formula are: $X_1, X_2, X_3, \neg X_1, \neg X_2, \neg X_3$.
- A literal included in a clause *satisfies* this clause.
- An *assignment* of a formula is a set of its literals, exactly one per variable.
- A 2-CNF formula *F* is satisfied by an assignment *P* if each clause of *F* is satisfied by at least one literal of *P*.
- A satisfying assignment of the formula in the above example is $\{X_1, \neg X_2 , X_3\}$.

Definition of the problem

Parameterized 2-CNF deletion problem (2-CNF-DEL)

Input: 2-CNF formula F and a parameter k.

Output: 'YES' if there is a set of at most k clauses whose removal makes the resulting formula satisfiable.

Two equivalent problems

1. <u>Input:</u> Graph *G,* parameter *k.*

   <u>Output:</u> 'YES' if *G* has a VC greater than the maximum matching of *G* by at most *k,* 'NO' otherwise.

   This is a more general parameterization of VC that by the output size

2. <u>Input:</u> CNF formula *F* (not necessary 2-CNF!), parameter *k.*

   <u>Output:</u> 'YES' if there are *k* variables such after their removal from *F*, the resulting formula is RENAMABLE HORN, 'NO' otherwise.

   This problem has applications in the design of SAT solvers.

Fixed-parameter tractability of the 2-CNF-DEL problem

- The question about the fixed-parameter tractability of the 2-CNF–DEL problem was first asked by Mahajan and Raman in JALG 31(2) pp. 335-354, 1999 (the preprint appeared two years earlier in ECCC 4(33), 1997).

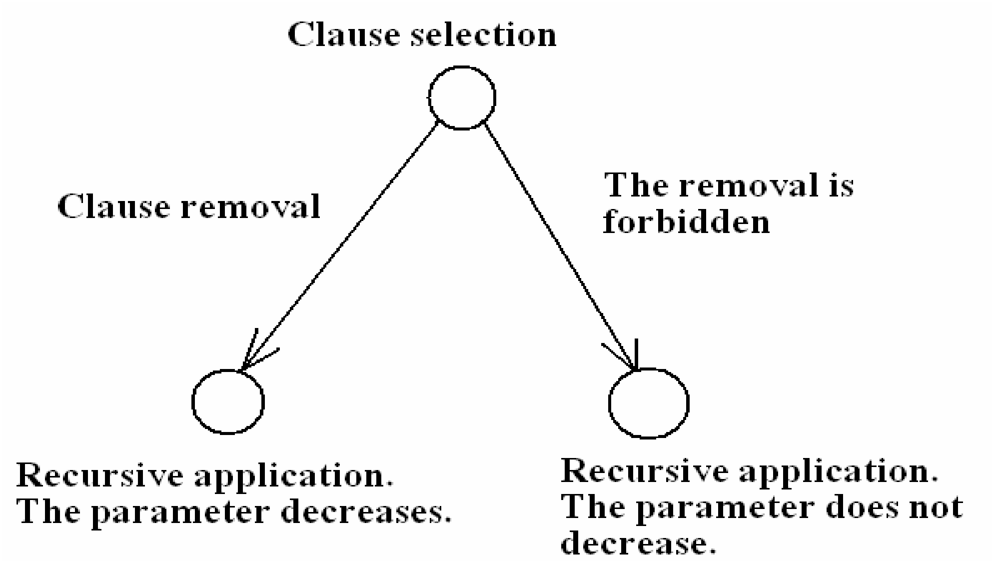
- Since then this question acquired reputation of one of the central challenges in the design of fixed parameter algorithms (see Niedermeier, "Invitation to fixed-parameter algorithms, volume 31, Oxford University press, page 277).

- The fixed-parameter tractability of this problem has been confirmed by Igor Razgon and Barry O'Sullivan in "Almost 2-SAT is fixed parameter tractable", ICALP 2008.

A fixed-parameter algorithm for the 2-CNF-DEL problem

The general idea of the algorithm

The basic procedure:



Problem with the second branch:

the parameter does not decrease

⇓

the height of the search tree is not guaranteed to be bounded by a function of *k*

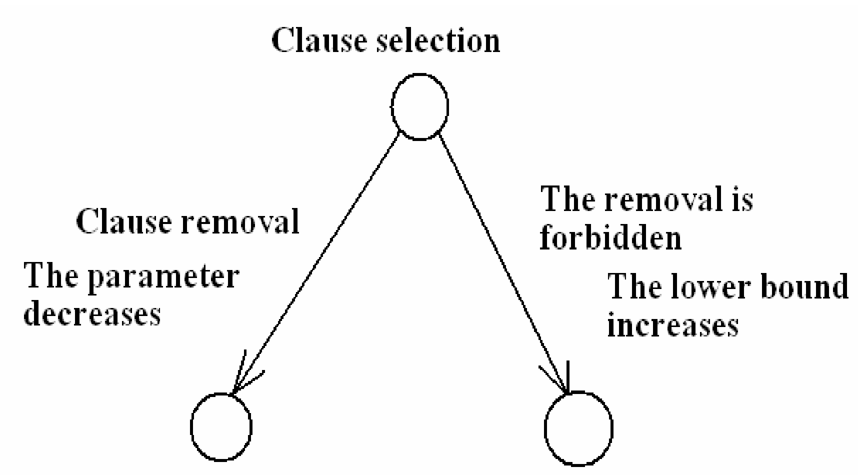
⇓

the algorithm is not necessary a fixed-parameter one.

The general idea of the algorithm (cont.)

A way to fix the problem: introducing a polynomially computable lower bound on the optimal solution size and recognizing 3 cases.

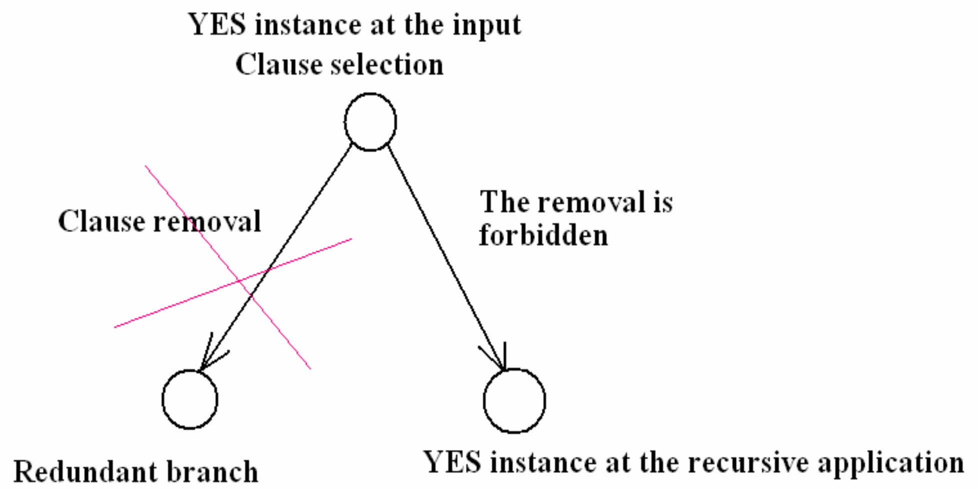
1. **The lower bound is greater than *k*.** ‘NO’ is returned immediately
2. **Forbidding the selected clause increases the lower bound.**



The gap between the parameter and the lower bound decreases on both branches → the height of the search tree depends on *k.*

The general idea of the algorithm (further cont.)

3. Forbidding the selected clause does not increase the lower bound.
Theorem:
<u>forbidding the selected clause does not increase optimal solution size</u> → the selected clause can be forbidden *without any branching.*



On each path from the root to a leaf of the search tree, the number of nodes with 2 or more children depends on k → the number of leaves of the search tree depends on k → the algorithm is a fixed-parameter one.

Auxiliary problem

The above strategy is in fact applied to an auxiliary problem, not to the 2-CNF-DEL problem directly. We define the auxiliary problem and show that if this problem is FPT then the 2-CNF-DEL problem is FPT as well.

A 2-CNF formula F is *satisfiable w.r.t.* a set of literals S if there is a satisfying assignment P of F such that S is a subset of P.

For example, $(X_1 \vee X_2)(\neg X_2 \vee \neg X_1)$ is satisfiable w.r.t. $\{X_1\}$

while $(\neg X_1 \vee X_2)(\neg X_2 \vee \neg X_1)$ is not.

Problem AUX

Input: (F,S,l,k), where F is a 2-CNF formula, S is a set of literals such that F is satisfiable w.r.t. S, l is a literal of F, k is the parameter.

Output: 'YES' if there is a set of at most k clauses of F whose removal makes resulting formula satisfiable w.r.t. $S \cup \{l\}$; 'NO' otherwise.

Auxiliary problem (cont.)

Theorem.

If problem AUX is FPT then the 2-CNF-DEL problem is FPT as well.

In the proof we show that the 2-CNF-DEL problem can be solved by making $O(3^k*m)$ calls to a procedure solving problem AUX, where m is the number of clauses of F.

We use a standard technique of proof called iterative compression See a survey paper: Huffner, Niedermeier, Wernicke "Techniques for practical fixed-parameter algorithms", The Computer Journal, 51(1), pages 7-25, 2008.

Problem AUX as a graph separation problem

We show that problem AUX can be represented as a separation problem on the implication graph of a 2-CNF formula

The implication graph $D(F)$ of a 2-CNF formula F is a directed graph whose set of vertices corresponds to the set of literals of F and (X_1,X_2) is an arc of $D(F)$ iff $(\neg X_1 \vee X_2)$ is a clause of F.

No bijection between clauses and arcs: an arc of $D(F)$ corresponds to exactly one clause of F while a clause of F generally corresponds to two different arcs of F.
In the above example, the additional arc associated with clause $(\neg X_1 \vee X_2)$ is $(\neg X_2, \neg X_1)$.

Let A and B be two sets of vertices of $D(F)$. A set C of clauses of F is an ***(A,B)*-separator** if removal of all the arcs corresponding to the clauses of C breaks all the paths from A to B in $D(F)$.

Problem AUX as a graph separation problem

Theorem

Let *(F,S,l,k)* be an instance of problem AUX.

This is a 'YES' instance if and only if *F* has an $(S \cup \{l\}, \{\neg l\})$ separator of size of size at most *k.*

The proof is similar to the proof of the unsatisfiability criterion of a 2-CNF formula (see, for example, "Computational Complexity" y Papadimitriou).

Polynomially computable lower bound

We introduce a polynomially computable lower bound on the size of *(S U {l}, {¬l})* separator. This lower bound is necessary for implementation of the general algorithmic scheme.

The smallest possible size of an *(S, {¬l})* separator is a lower bound on the size of *(S U {l}, {¬l})* separator.

Theorem.
A smallest *(S, {¬l})* separator is polynomially computable.

Proof sketch.

S is satisfiable w.r.t. *F* → two arcs corresponding to the same clause cannot be simultaneously reachable from *S* → there is a bijection between the edges reachable from *S* and the corresponding clauses → the smallest *(S, {¬l})* separator can be computed by a network flow algorithm.

Clauses forbidden for removal

We describe instances of problem AUX to which the algorithm is recursively applied if the selected clauses is forbidden to be removed.

Assume that *(F,S,l,k)* is a 'YES' instance of problem AUX and clause $(X_1 \vee X_2)$ is forbidden to be removed. Then the resulting formula must be satisfiable either w.r.t. $S \cup \{X_1\}$ or w.r.t. $S \cup \{X_2\}$.

Therefore forbidding the clause generally requires two recursive applications: one to $(F, S \cup \{X_1\}, l, k)$, the other to $(F, S \cup \{X_2\}, l, k)$.

Remark. The algorithm is applied to instance $(F, S \cup \{X_i\}, l, k)$ only if *F* is satisfiable w.r.t. $S \cup \{Xi\}$. Otherwise, the respective branch is omitted.

Neutral literals

A literal *l'* of a 2-CNF formula *F* is a neutral literal of instance *(F,S,l,k)* of problem AUX if the size of a smallest *(S, {¬l})*- separator is the same as the size of a smallest *(S U {l'}, {¬l})*-separator.

Theorem.

If *l'* is a neutral literal of *(F,S,l,k)* and *(F,S,l,k)* is a 'YES' instance of problem AUX then *(F,S U{l'},l,k)* is a 'YES' instance as well.

Corollary.

If $(X_1 \vee X_2)$ is a clause of *F* and X_1 is a neutral literal of *(F,S,l,k)* then without any branching clause $(X_1 \vee X_2)$ can be forbidden and algorithm can recursively apply to $(F,S \cup \{X_1\},l,k)$.

The algorithm

SolveAUX(F,S,l,k)

If F is satisfiable w.r.t. $S \cup \{l\}$ **then** return 'YES'

Let LB be the size of a smallest $(S,\{\neg l\})$-separator

If $LB>k$ **then** return 'NO'

Select a clause $C=(X_1 \vee X_2)$

If some X_i is a neutral literal of (F,S,l,k) **then**

Return $SolveAUX(F,S \cup \{X_i\},l,k)$

If $SolveAUX(F \setminus C,S,l,k-1)$ returns 'YES' **or**

$SolveAUX(F,S \cup \{X_1\},l,k)$ returns 'YES' **or**

$SolveAUX(F,S \cup \{X_2\},l,k)$ returns 'YES' **then**

Return 'YES'

Else return 'NO'

Runtime evaluation of *SolveAUX*

Theorem.

Each path from the root to a leaf in the search tree contains at most $2k$ branching nodes.

Proof sketch.

- Each node is associated with measure $2k-LB$.
- The measure of the root is at most $2k$.
- A node with measure 0 is a leaf.
- The measure of any child of a branching node is smaller than the measure of this node itself.

Clause removal decreases the parameter by 1 and decreases LB by at most 1 $\rightarrow$
the measure of the corresponding child is at most $2(k-1)-(LB-1) < 2k-LB$.
Clause forbidding increases LB by 1 $\rightarrow$ the respective measure becomes $2k-(LB+1) < 2k-LB$.

Runtime evaluation of SolveAUX (cont.)

Corollary.

The search tree has at most 9^k leaves.

This corollary immediately follows from the last theorem, taking into account that each branching node has at most *3* branches.

A more careful evaluation allows to prove that the number of leaves of the search tree can be bounded by 5^k. Taking into account the polynomial factors, the runtime of *SolveAUX(F,S,l,k)* is $O(5^k km^2)$, where m is the number of clauses of *F.* Taking into account that $O(3^k m)$ calls to *SolveAUX* are needed to solve the 2-CNF-DEL problem,we get the final theorem.

Theorem.

2-CNF-DEL problem can be solved in time $O(15^k km^3)$

Summary

- Fixed-parameter algorithms are techniques of coping with NP-hardness that are useful in situations where problems are associated with parameters that are very small in practice. Problems that can be solved by fixed-parameter algorithms are called fixed-parameter tractable (FPT).

- 2-CNF deletion problem asks whether at most *k* clauses can be removed from a 2-CNF formula to make it satisfiable.
The are a number of reasons while it is worthwhile to solve this problem by a fixed-parameter algorithm. Nevertheless, the status of fixed-parameter tractability of this problem had been open for more than 10 years.

- We have shown that this problem is FPT.