Coloring perfect graphs by contraction

Benjamin Lévêque

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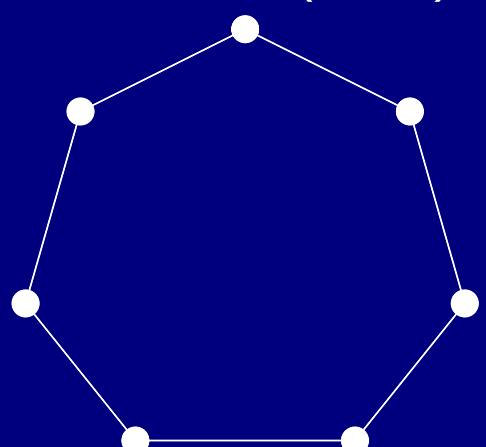
Odd hole (2k+1)



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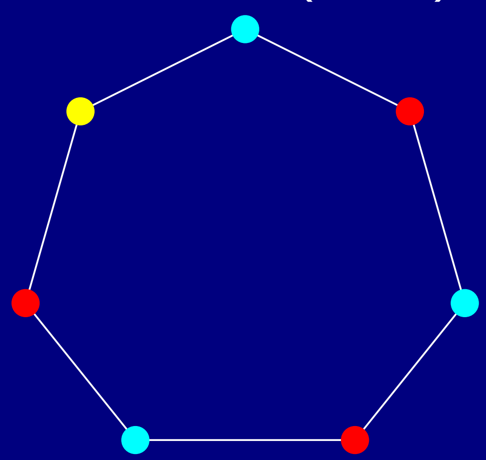

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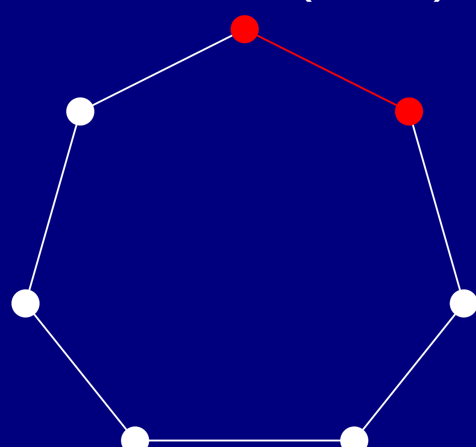

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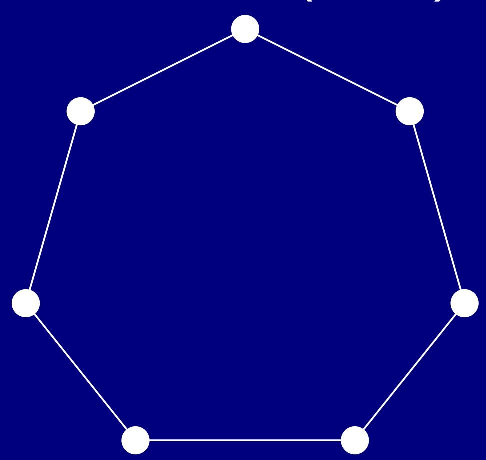

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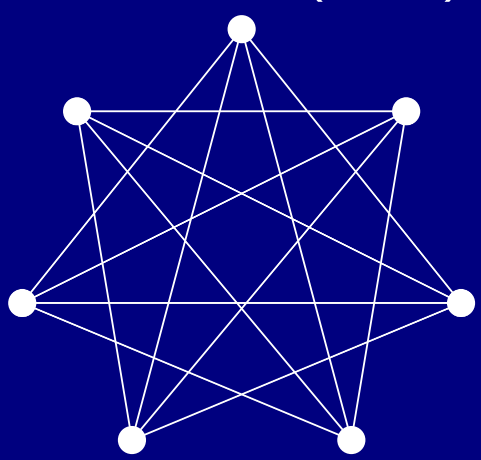

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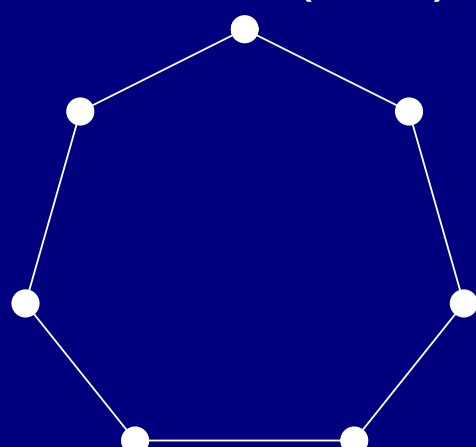

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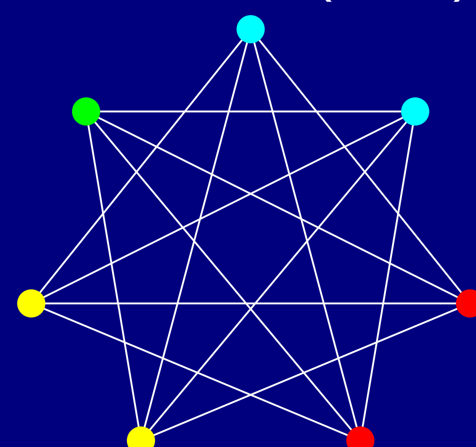

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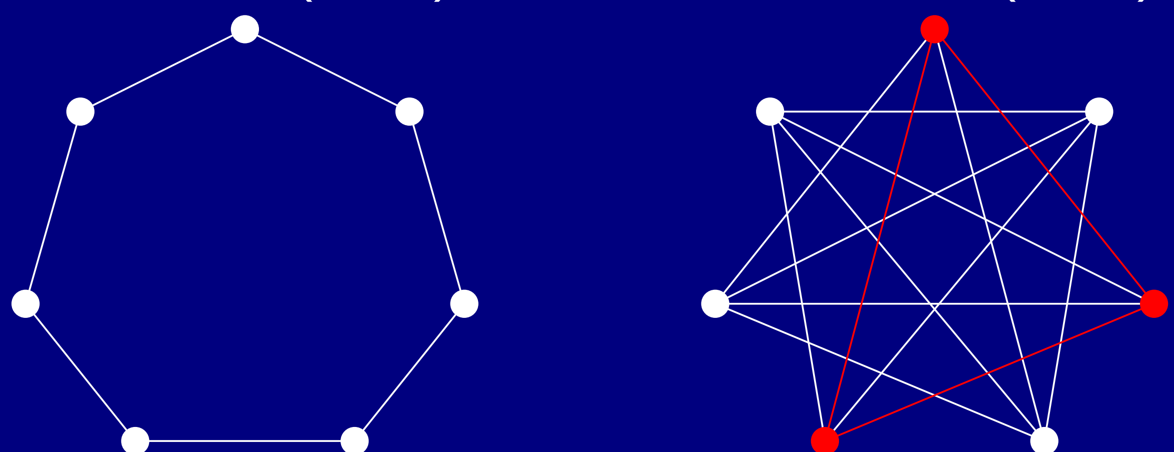

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Perfect graphs (Berge - 1960)

For every induced subgraph H of G : $\chi(H) = \omega(H)$

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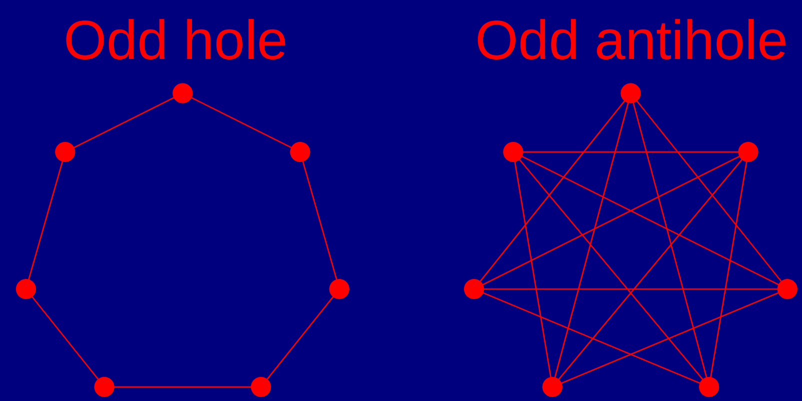

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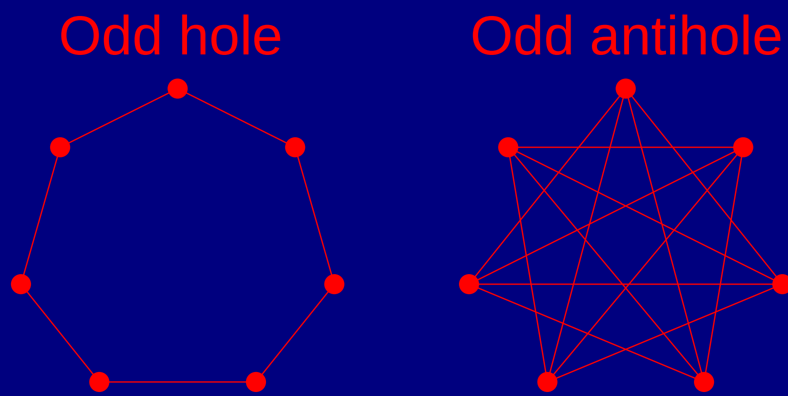

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G is perfect iff it contains no odd hole and no odd antihole

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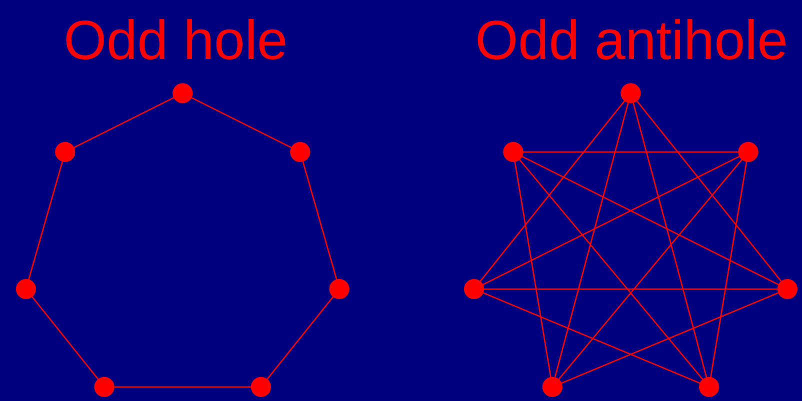

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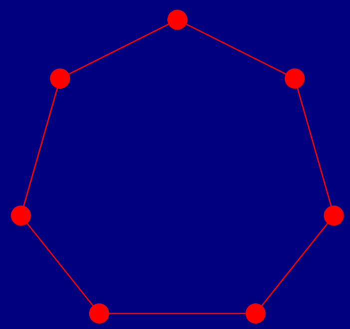
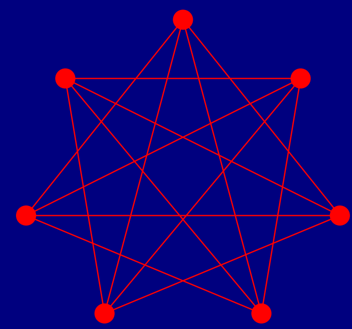


Odd antihole



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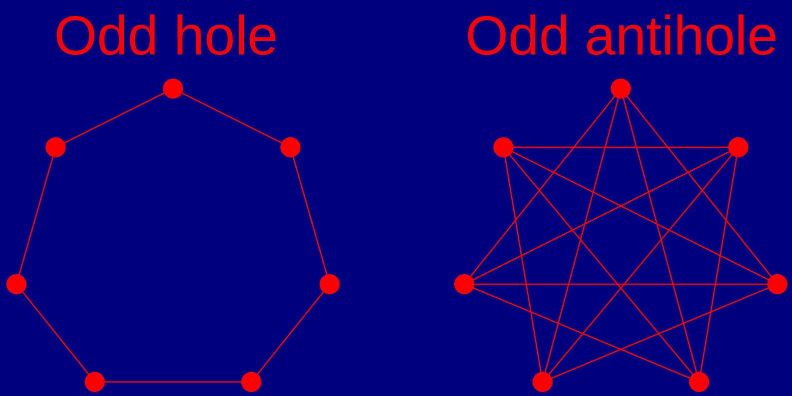

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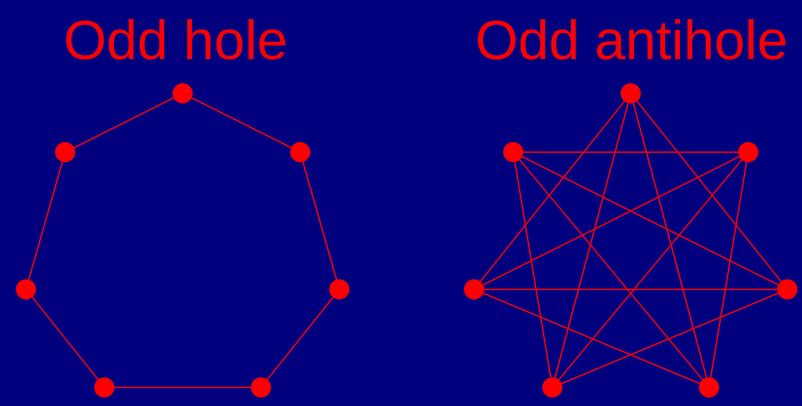

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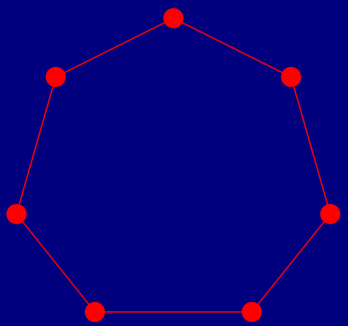

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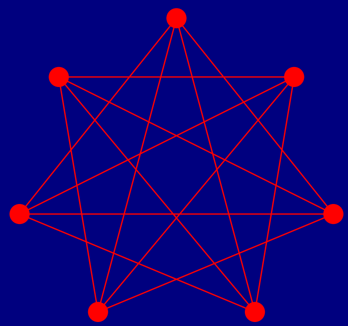

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Purely combinatorial coloring algorithm ?

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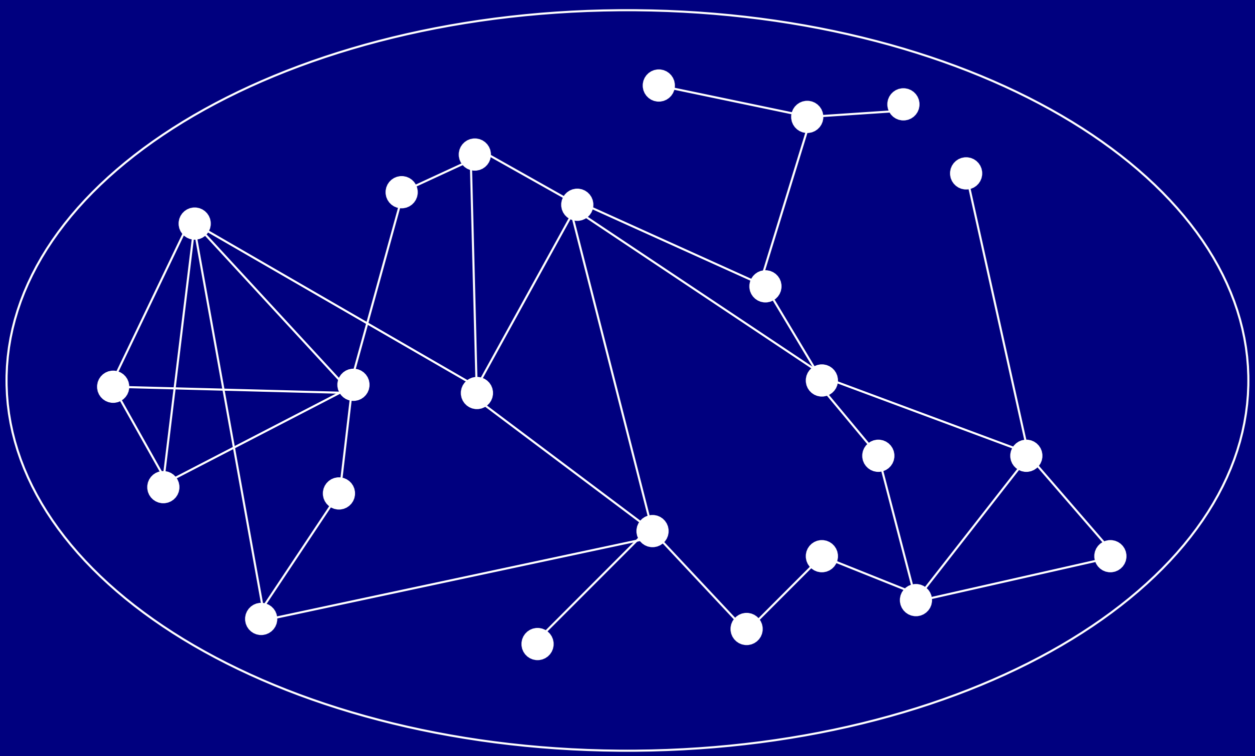

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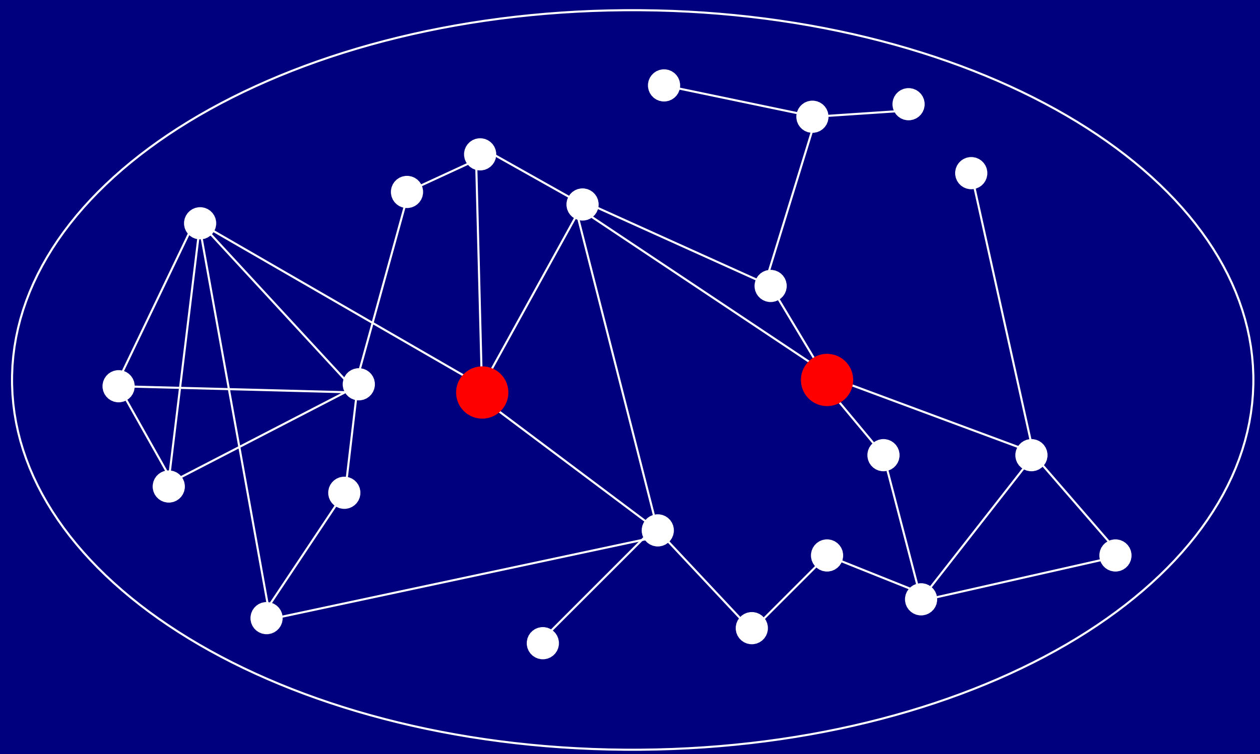

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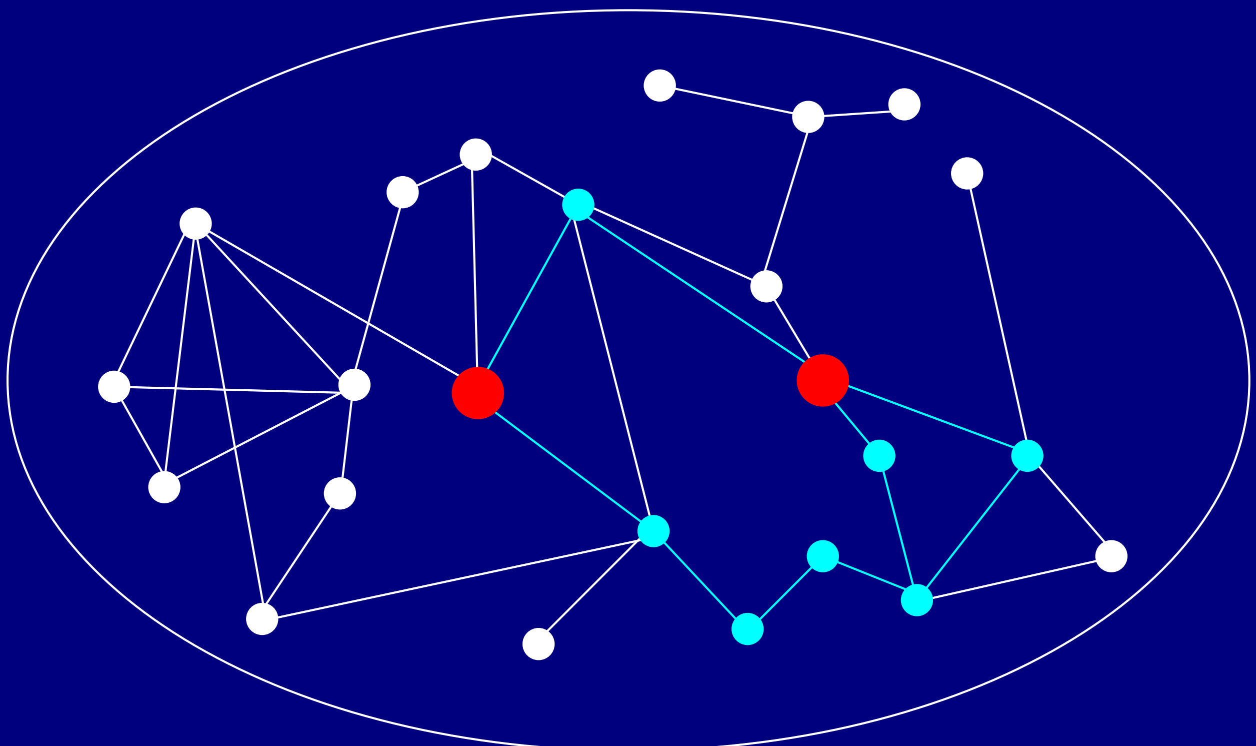

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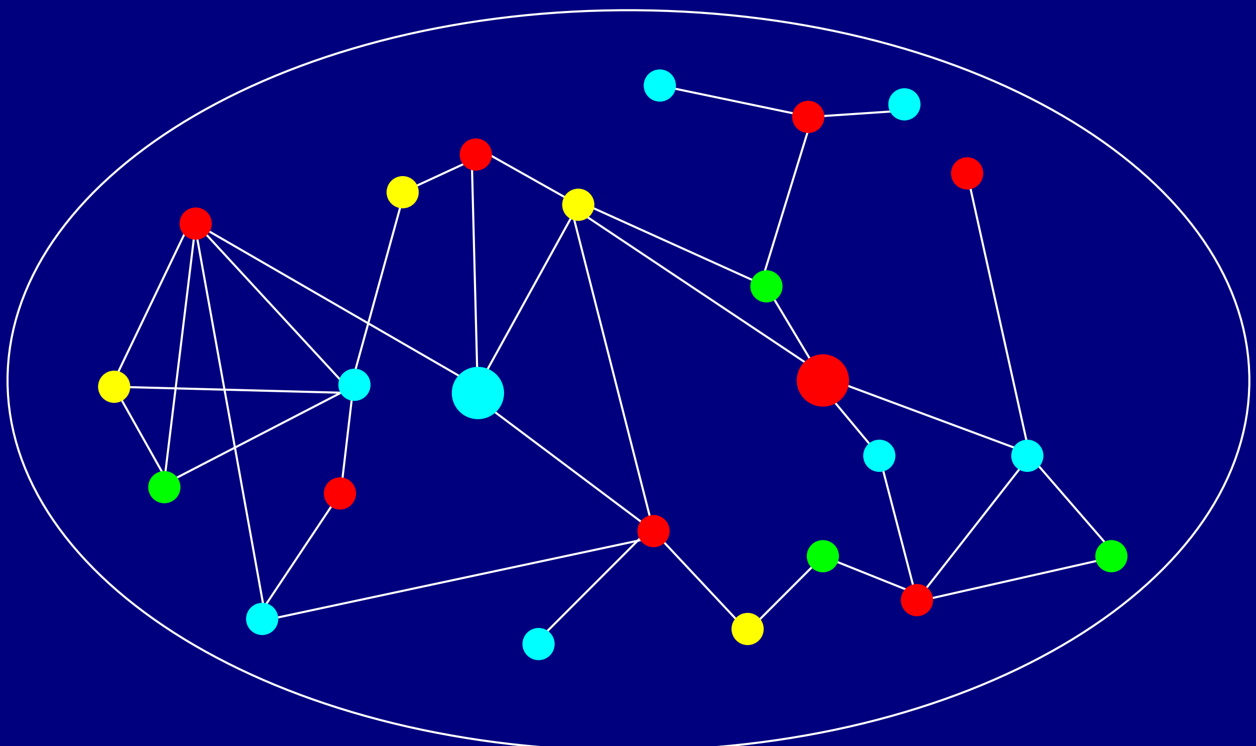

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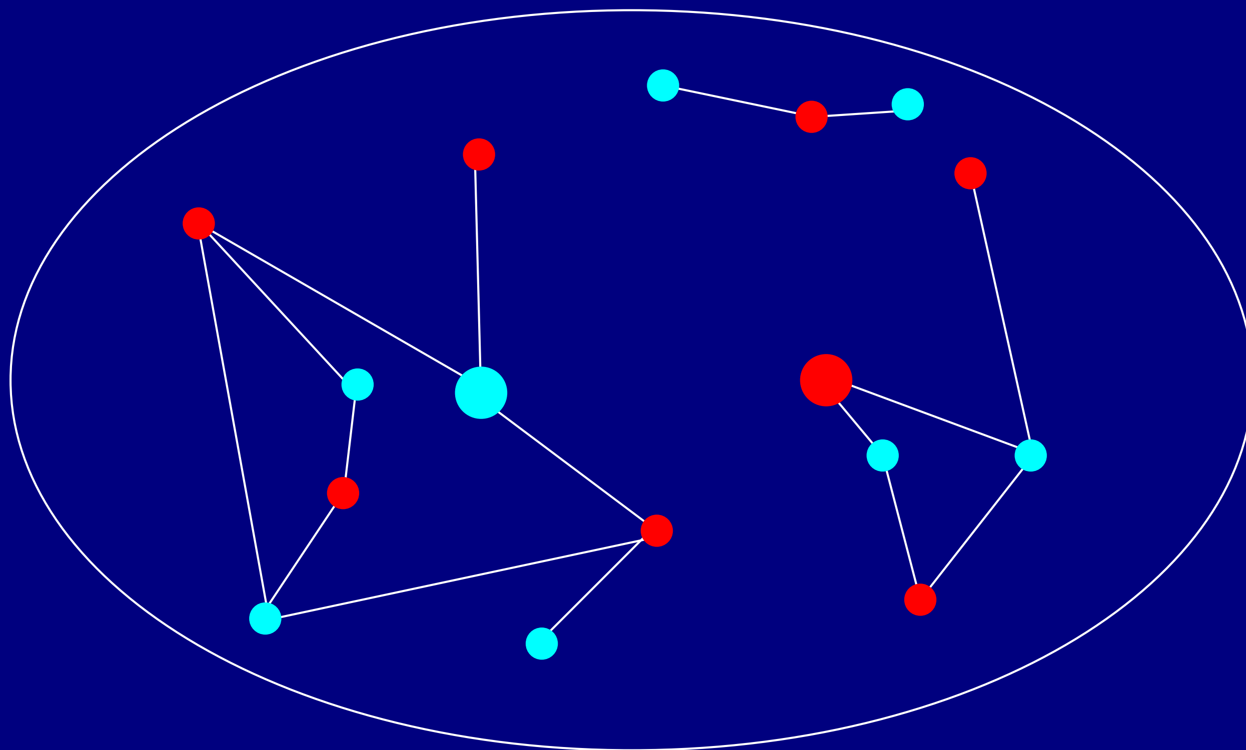

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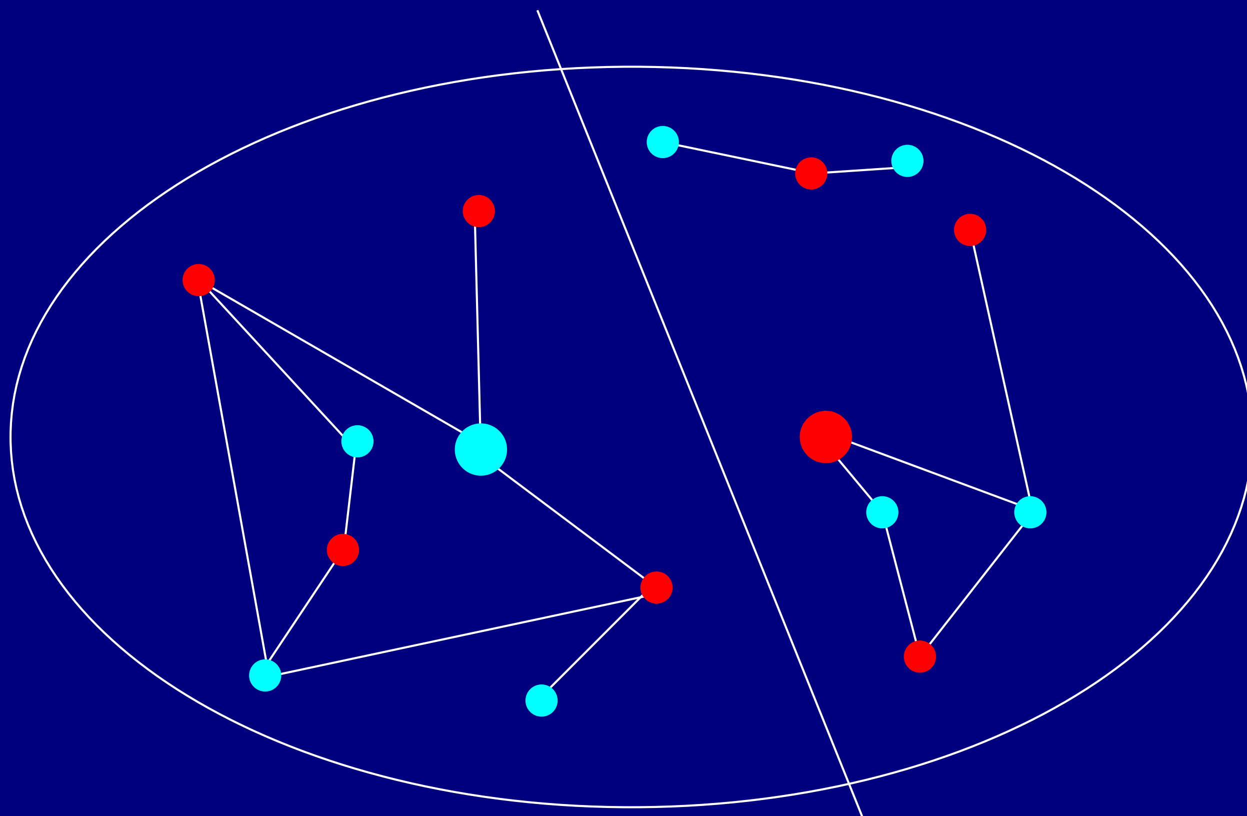

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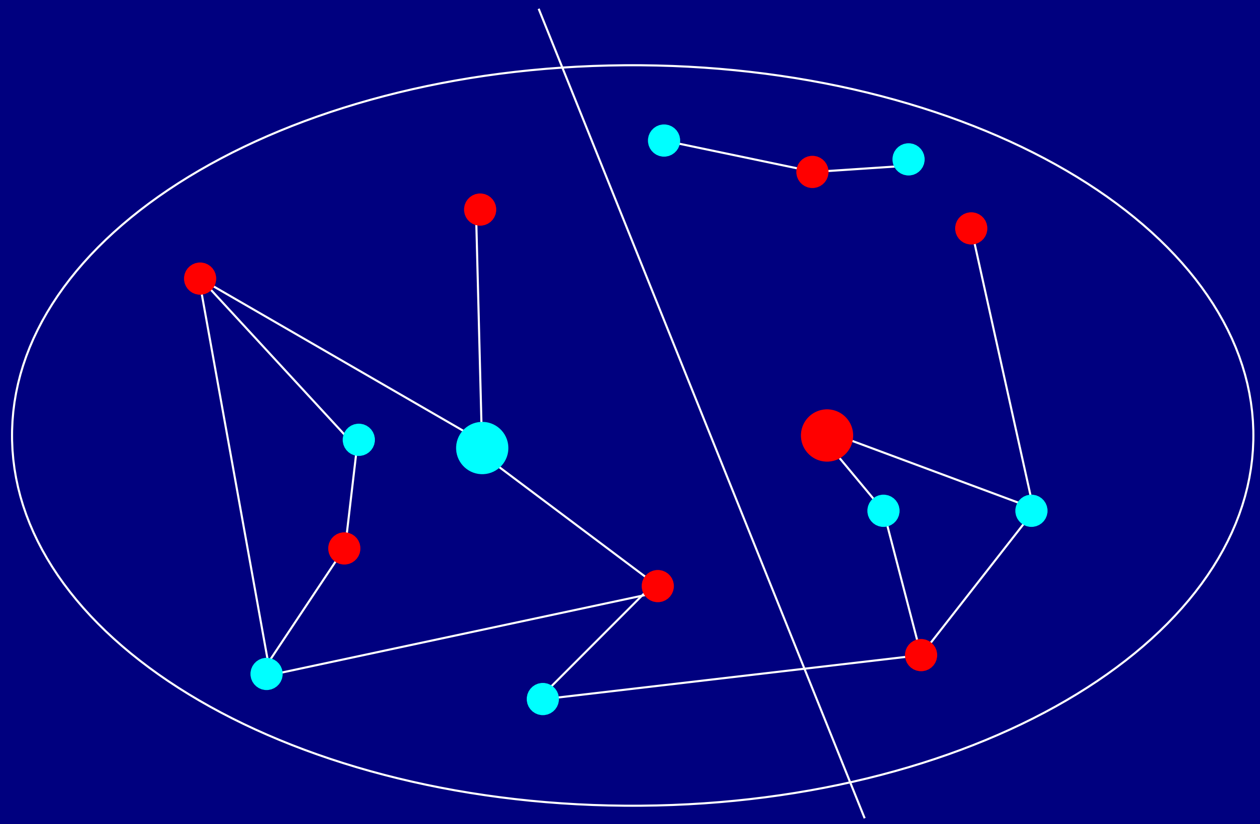

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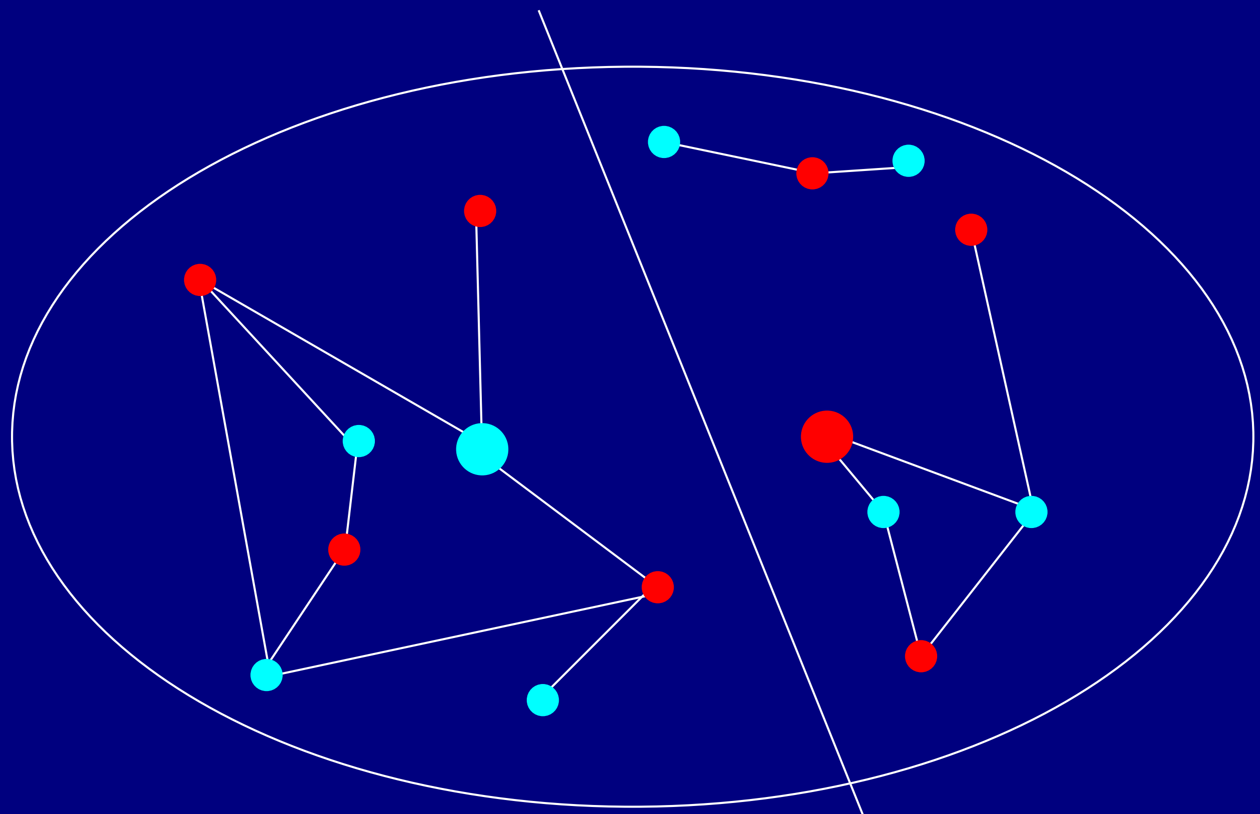

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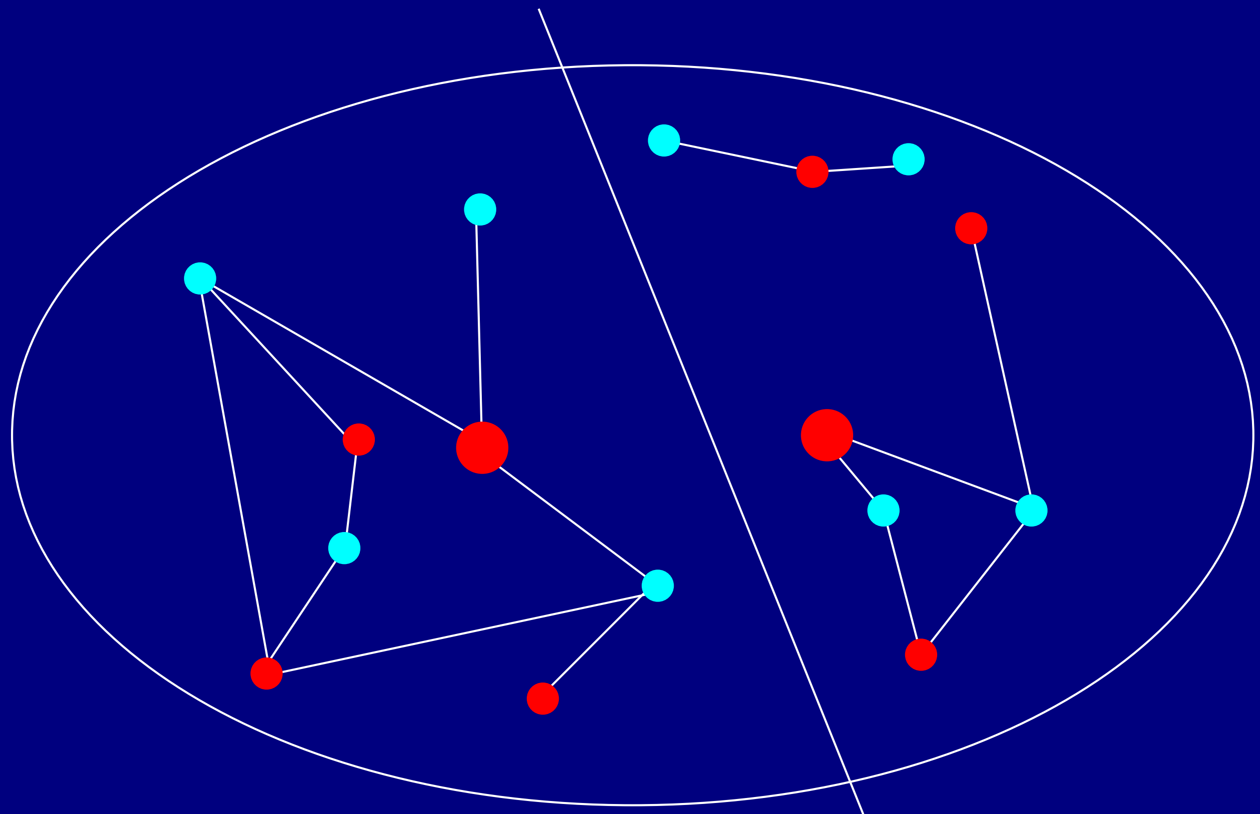

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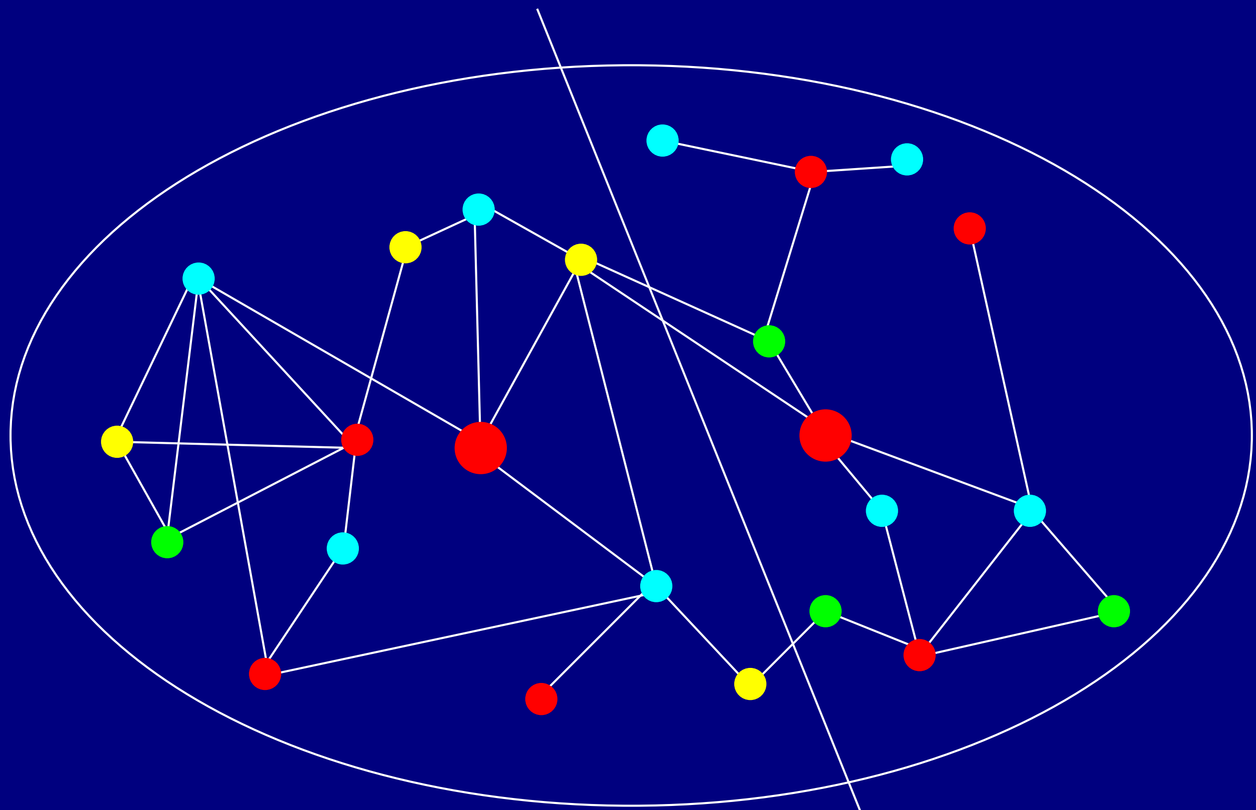

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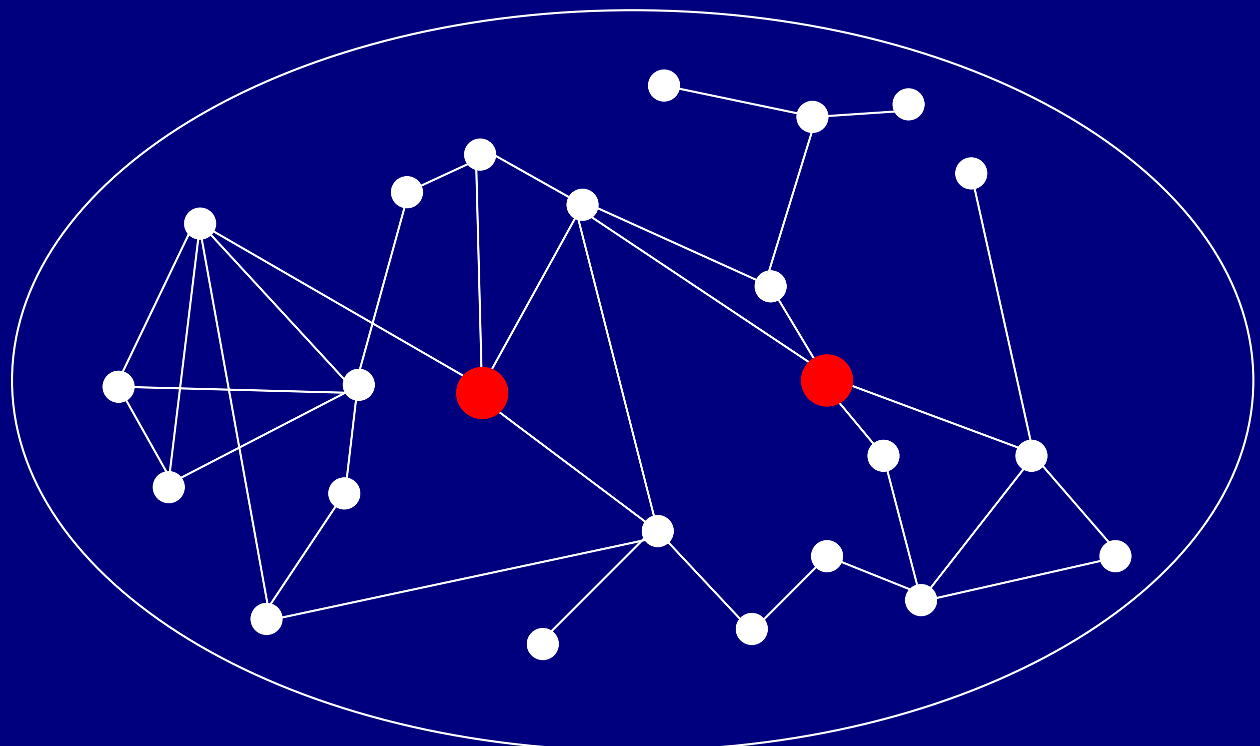

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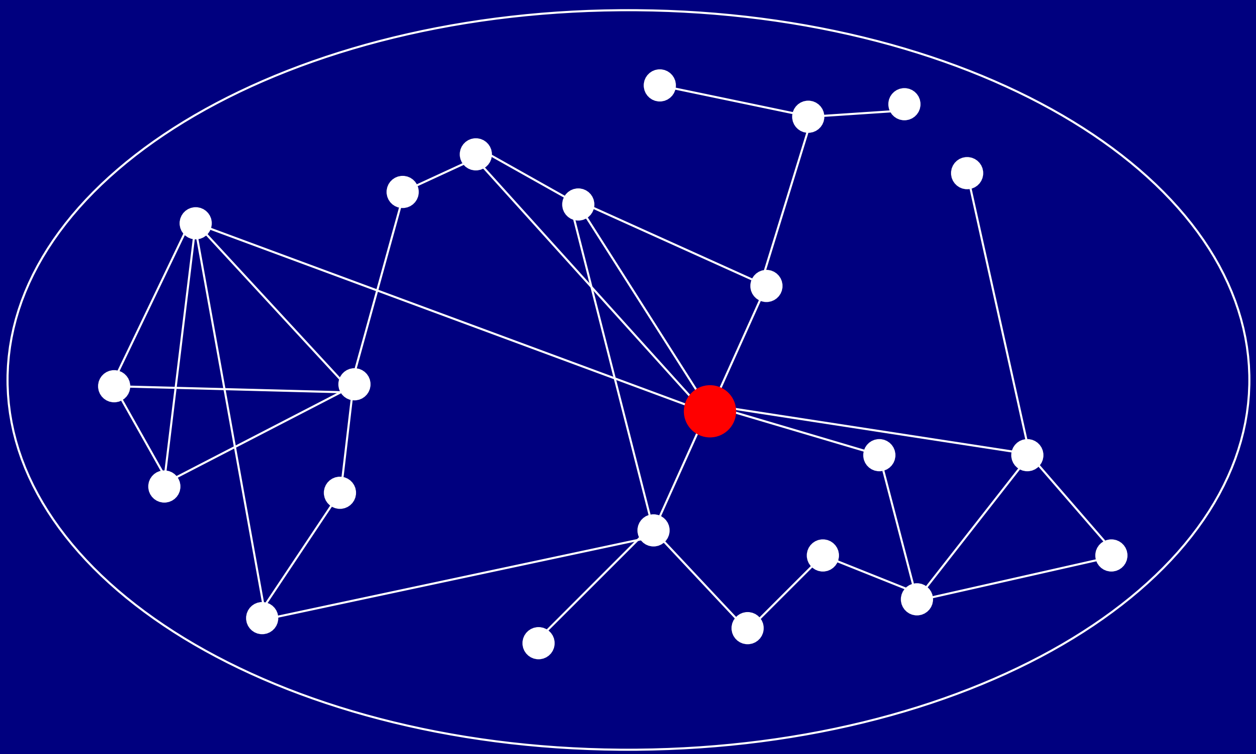

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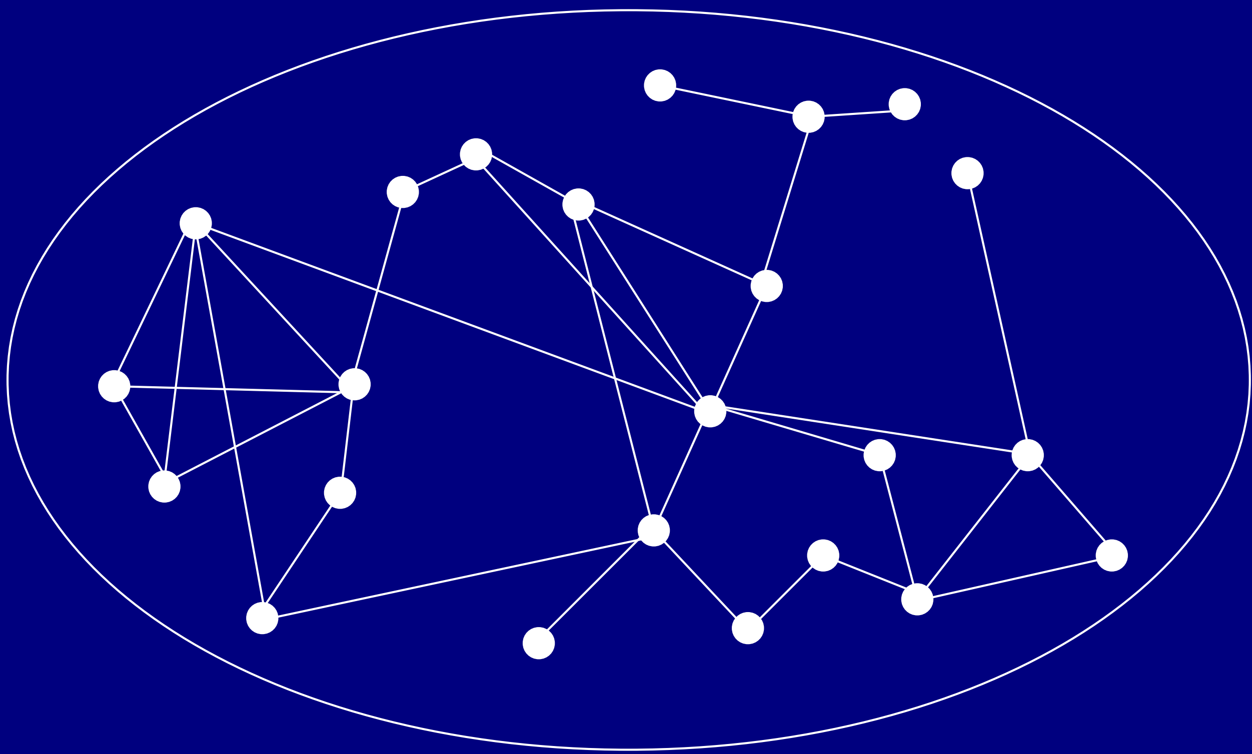

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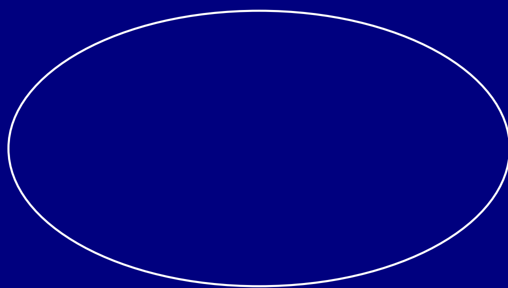

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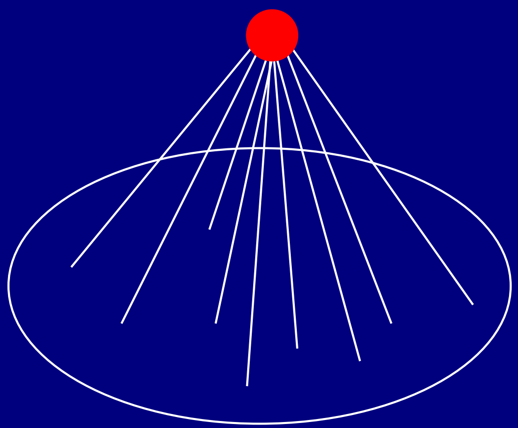

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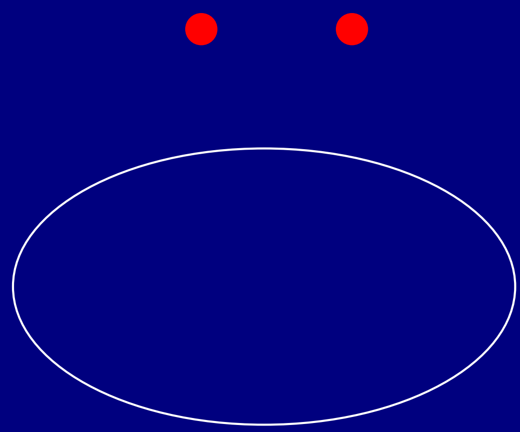

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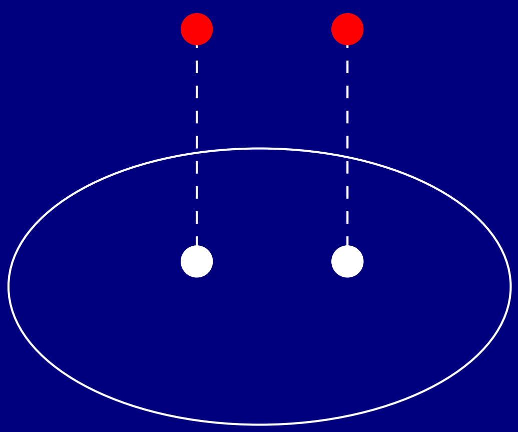

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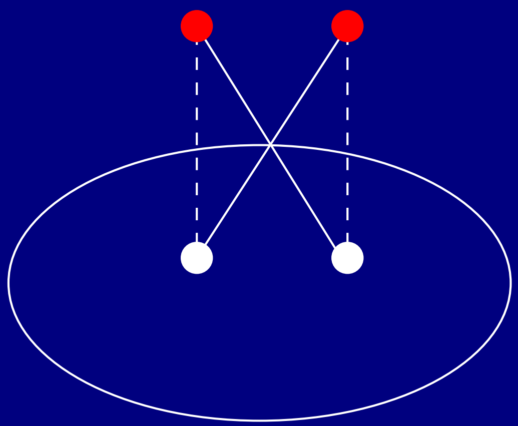

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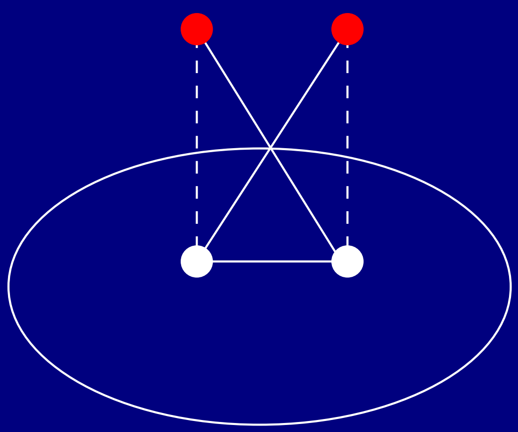

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Conjecture (Everett, Reed - 1993)
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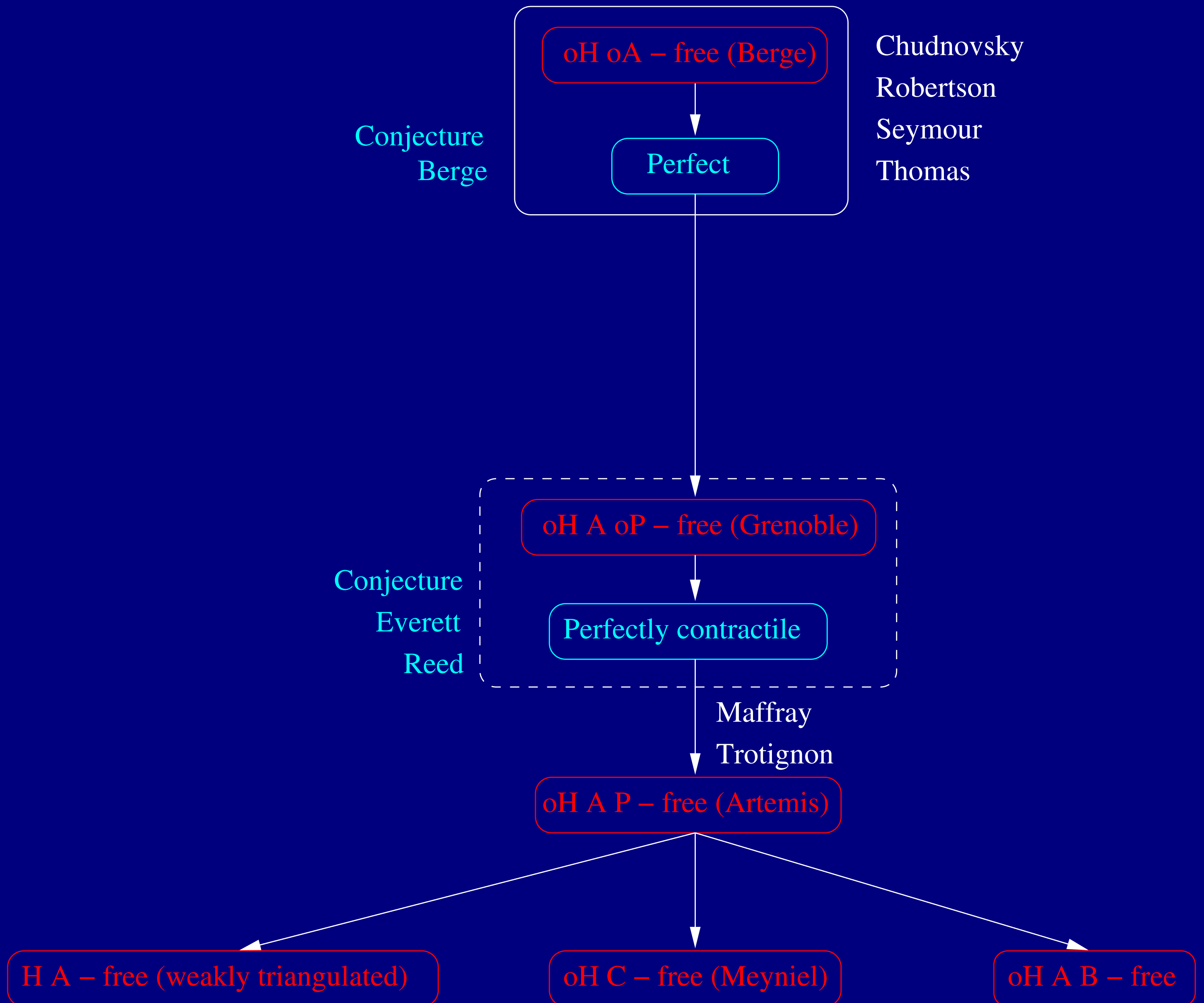
oH oA – free (Berge)
Perfect
Conjecture
Berge
Chudnovsky
Robertson
Seymour
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Perfectly contractile
Conjecture
Everett
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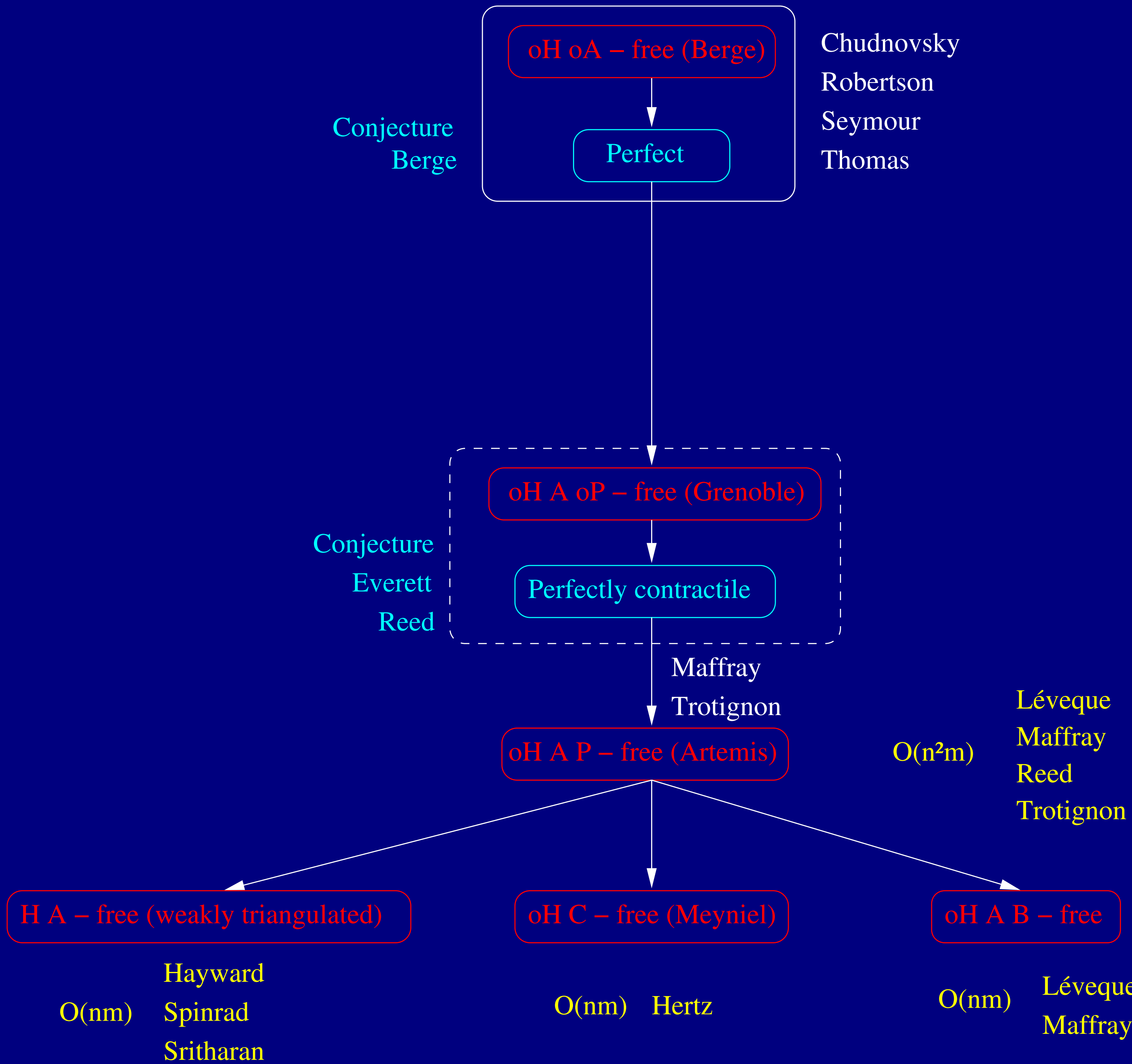
Maffray
Trotignon
oH A P – free (Artemis)
H A – free (weakly triangulated)
oH C – free (Meyniel)
oH A B – free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O(nm)
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Spinrad
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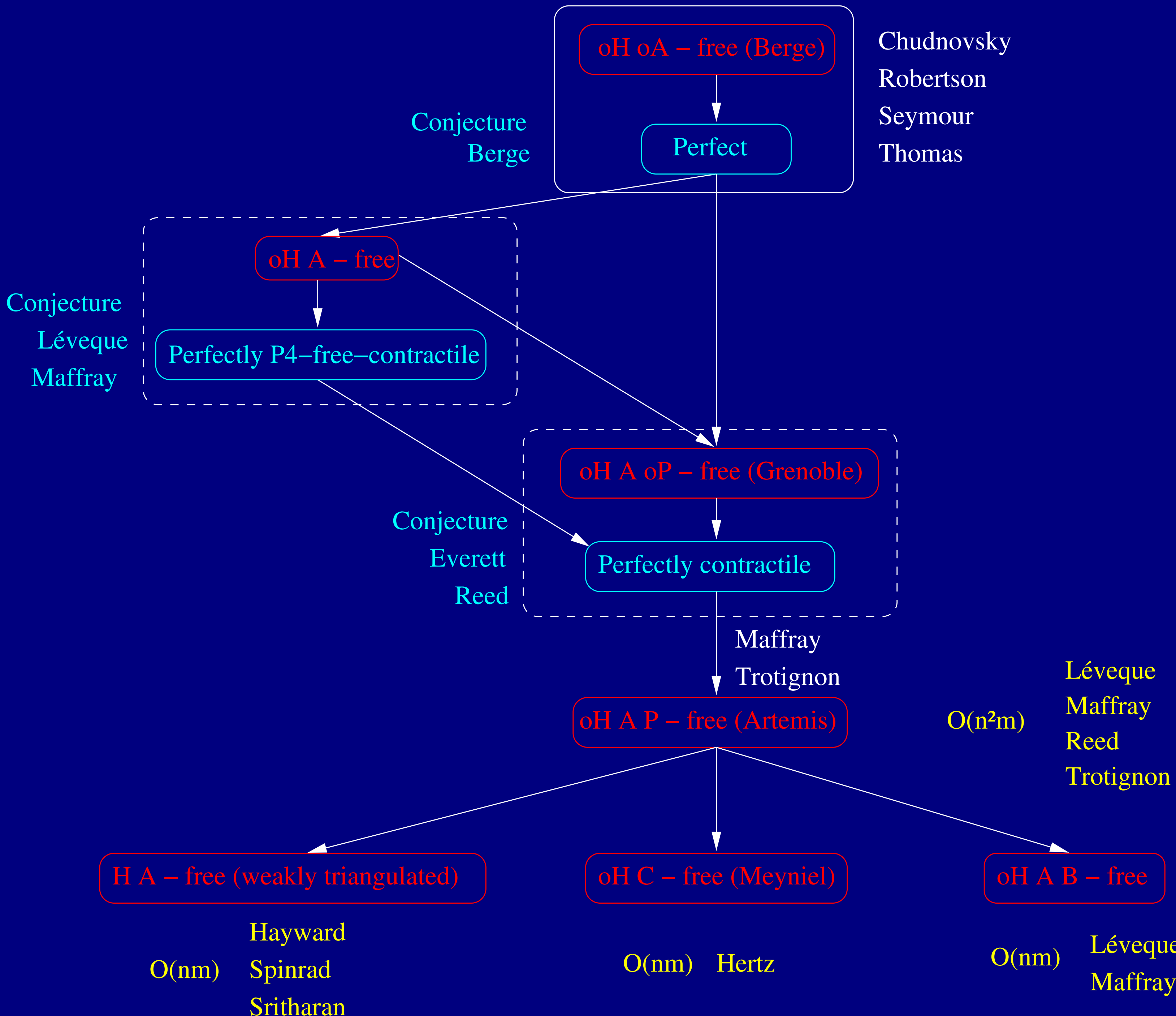
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Bull-free Artemis graphs

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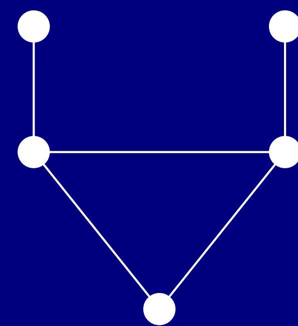

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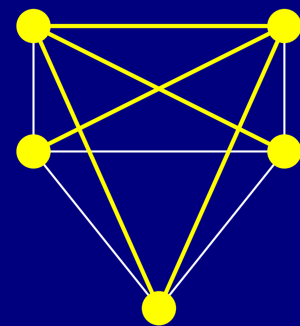
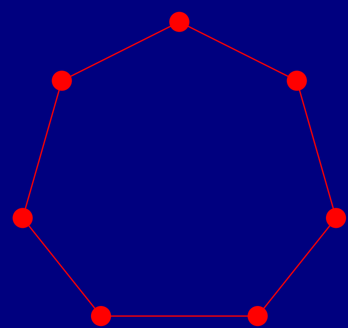


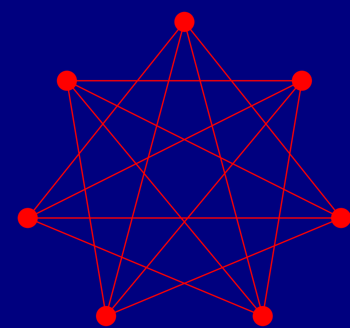
Bull-free Artemis graphs

G is bull-free Artemis iff it contains no odd hole, no antihole and no bull

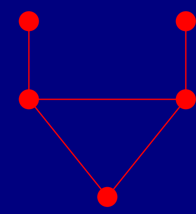
Odd hole



Antihole



Bull



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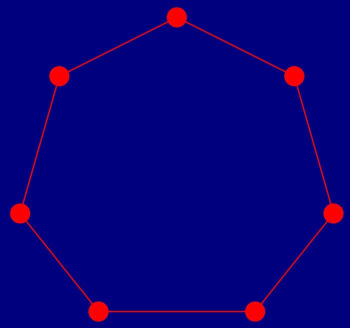
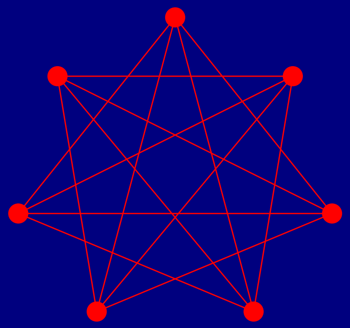
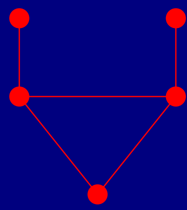


Antihole



Bull



Algorithm Cosine (Hertz - 1990)

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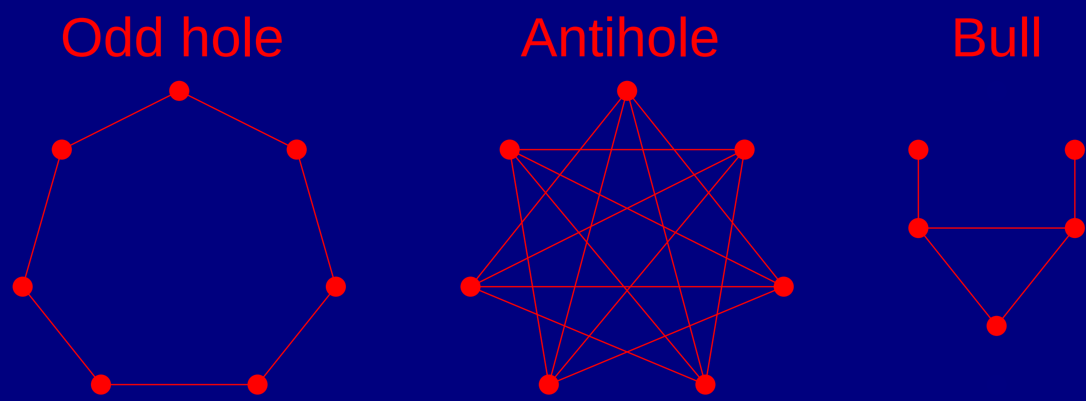

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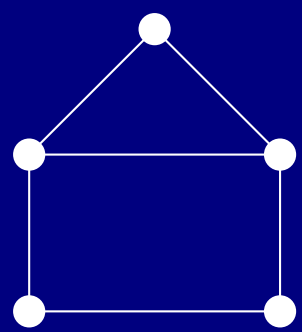

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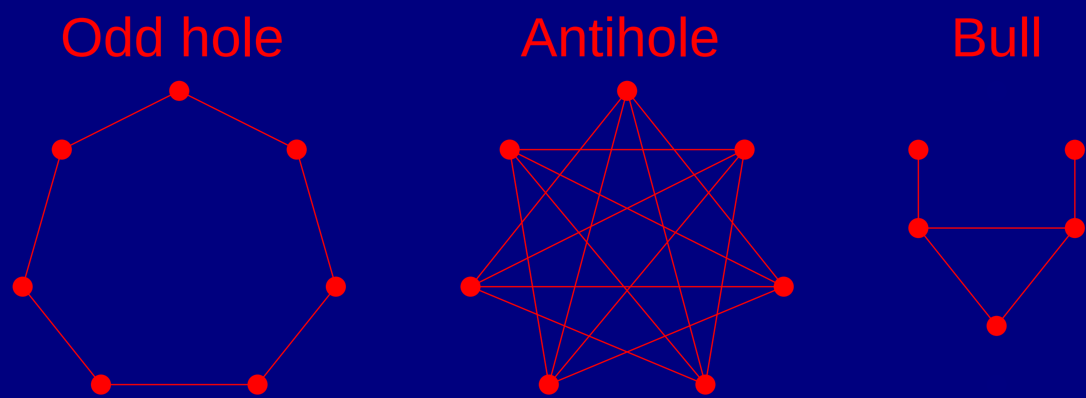

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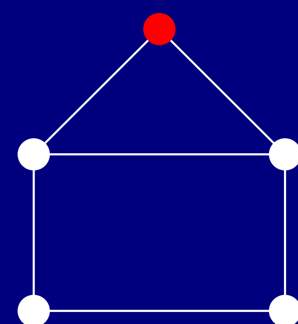

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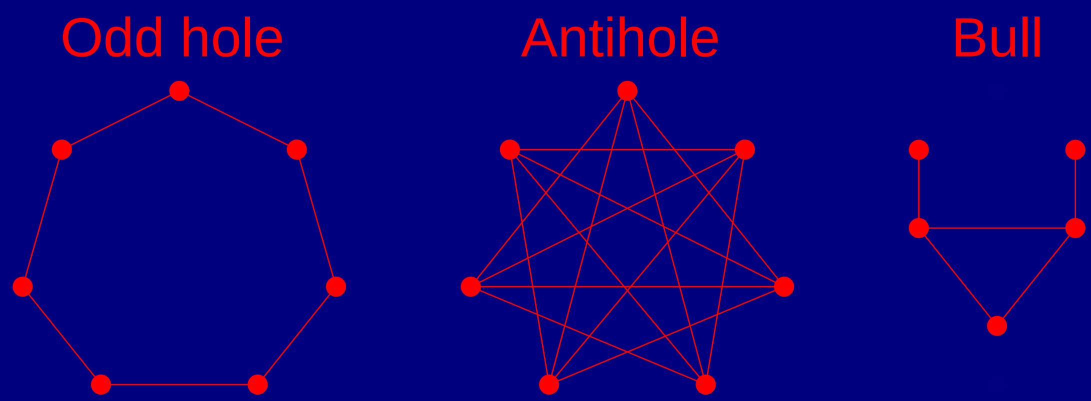

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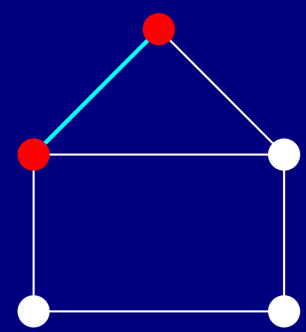

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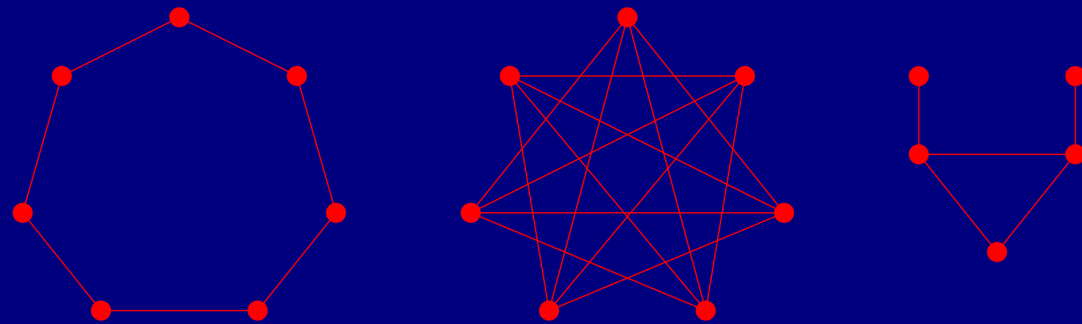
Odd hole　　Antihole　　Bull



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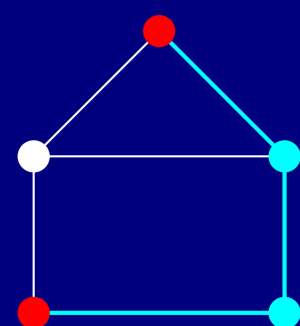

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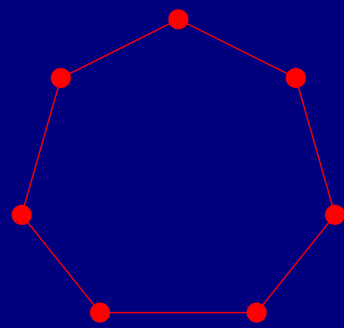

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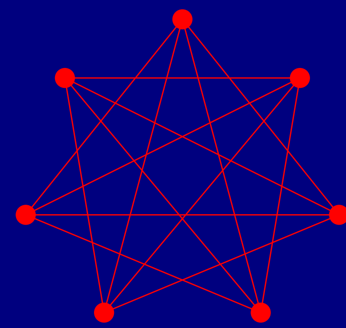

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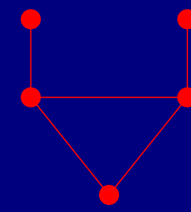

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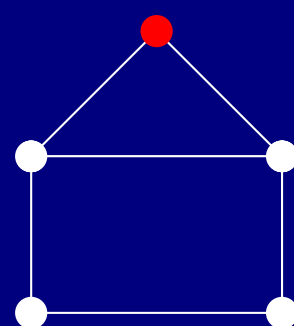

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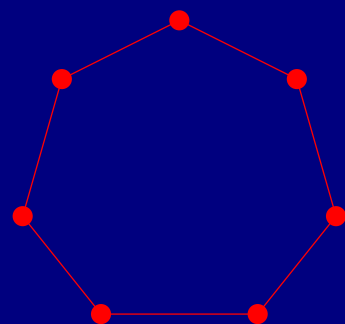

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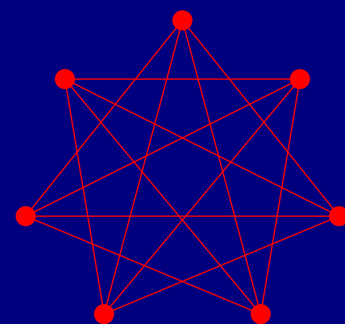

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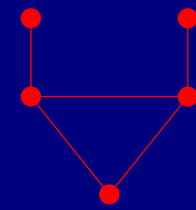

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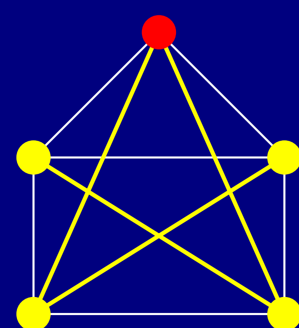

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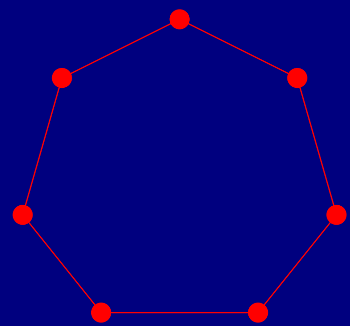

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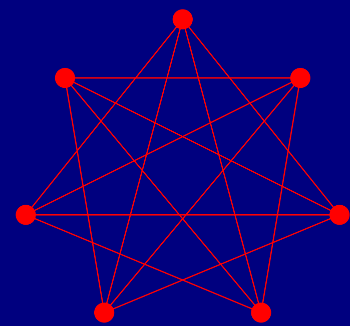

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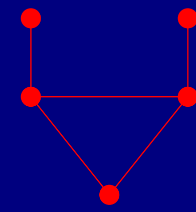

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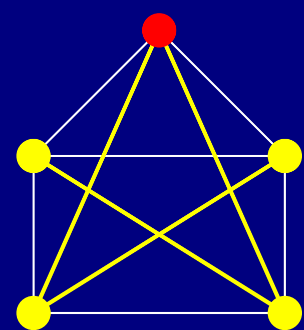

Theorem (Lévêque, Maffray - 2007)
Every hole-free bull-free graph has a vertex that is not the middle of a P_5

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ALGORITHM LEXBFS (Rose, Tarjan, Lueker 1976)

Input: A graph G with n vertices.
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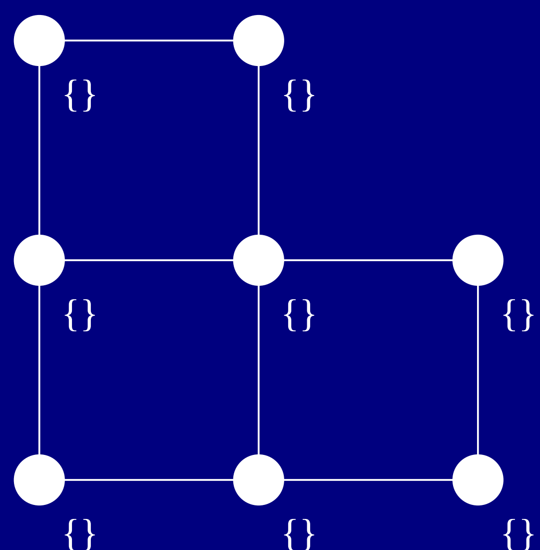

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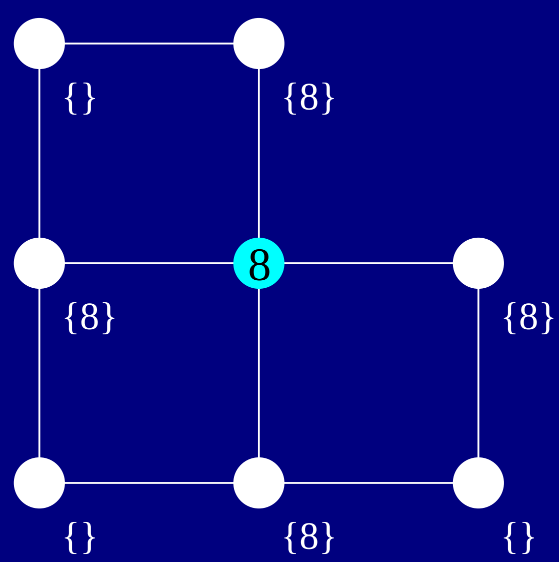

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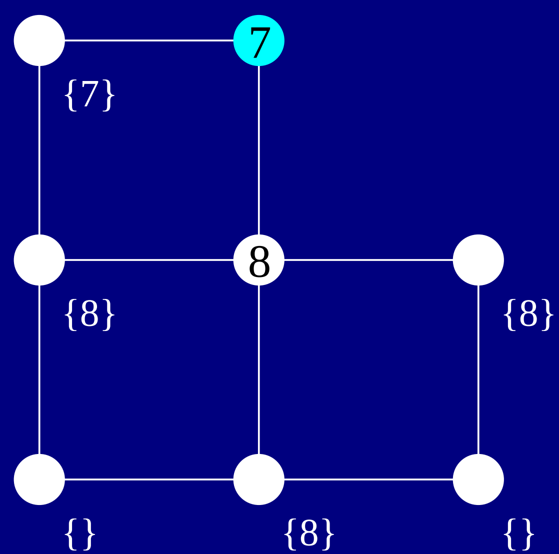

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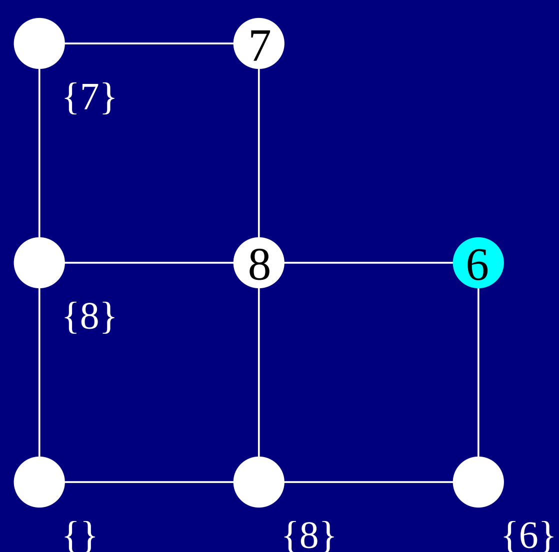

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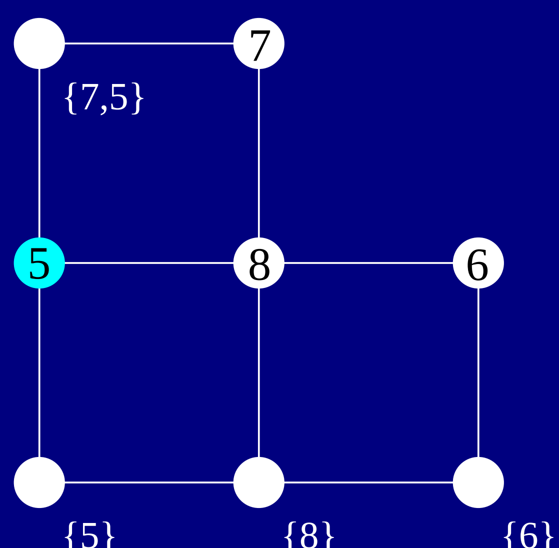

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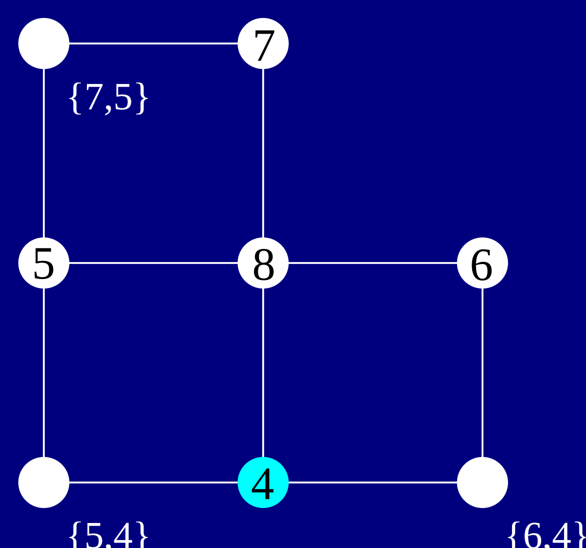

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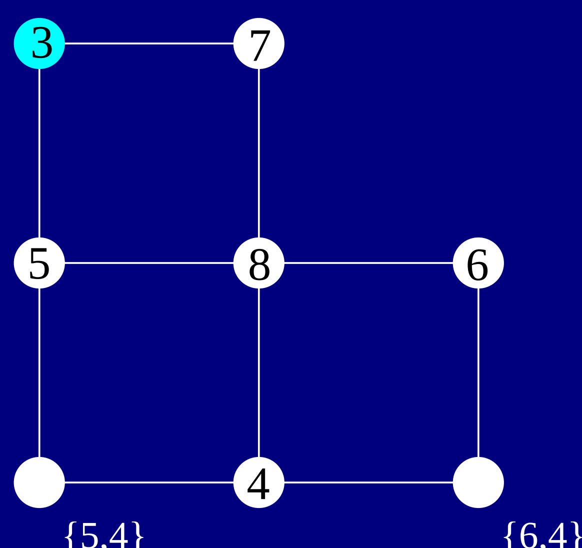

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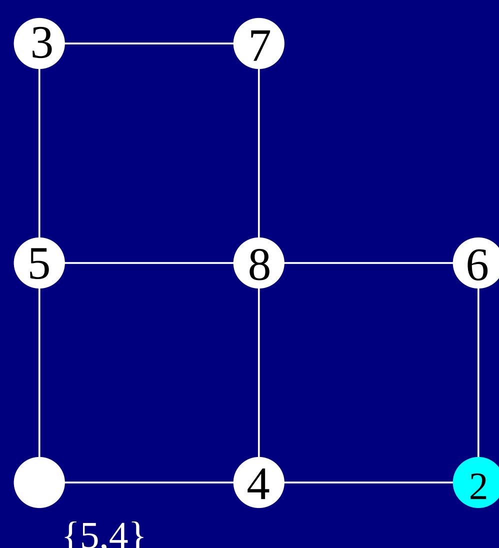

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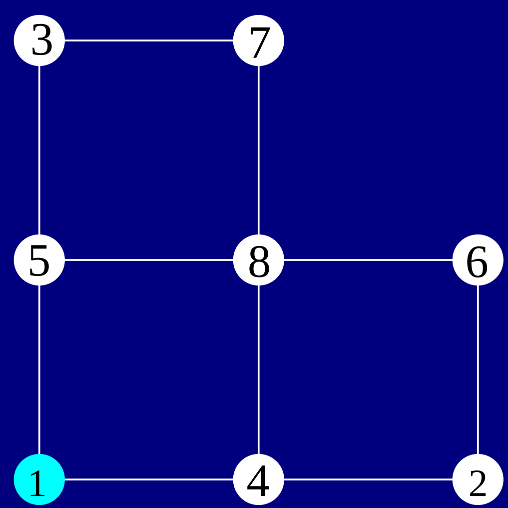

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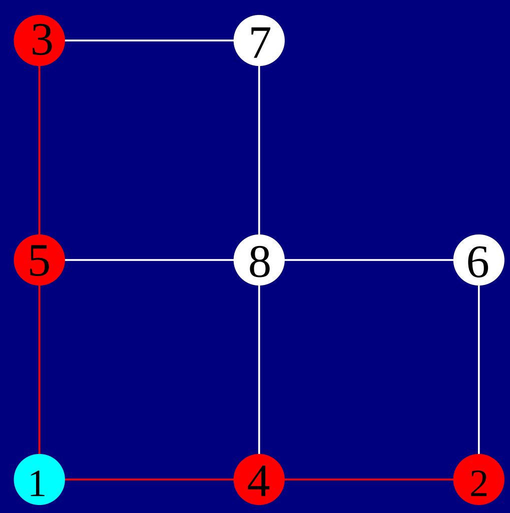

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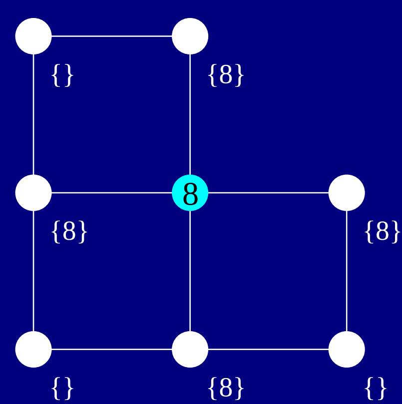

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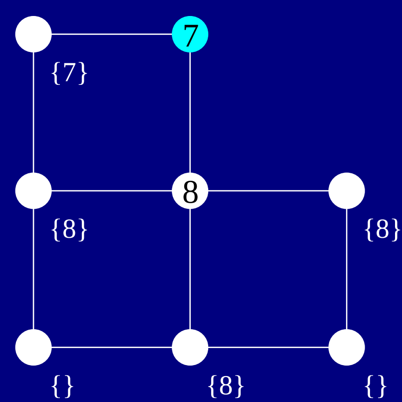

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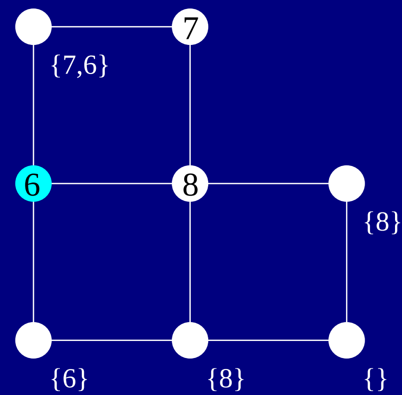

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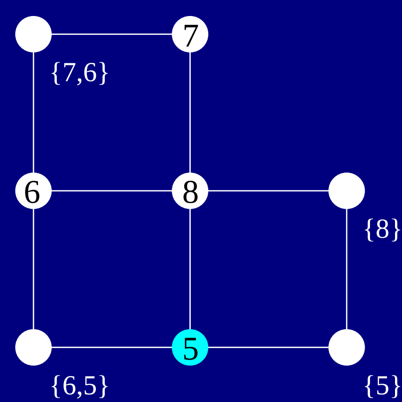

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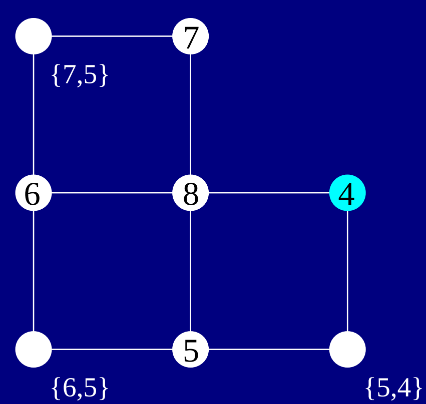

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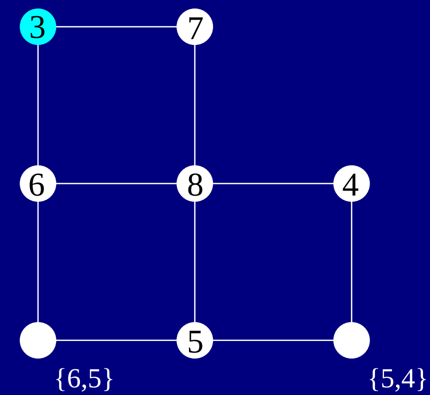

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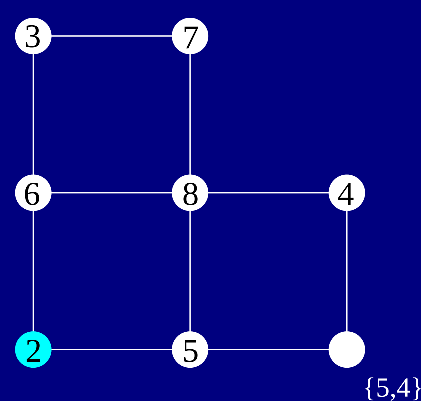

Algorithm LexBFS*

Input: A graph G with n vertices.
Output: An ordering σ on the vertices of G.

Initialization: For every vertex a of G, set $L(a) := \emptyset$;

General step: For $i = n, \ldots, 1$ do:
- Let A be the set of unnumbered vertices whose label is maximum.
- Let U be the other unnumbered vertices.
- Until $|U| = 0$ do:
  - Select a vertex $u \in U$ for which $L(u) \setminus L(A)$ is maximum.
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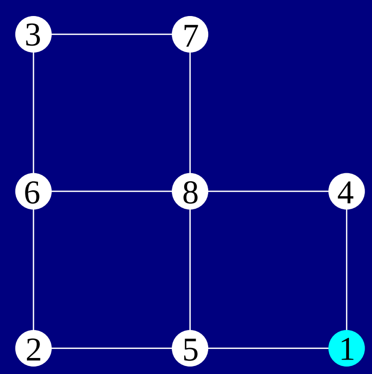

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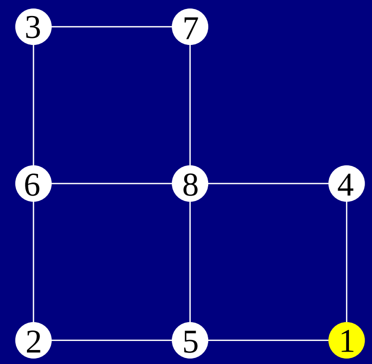

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LexBFS $\mathcal{O}(n+m) \longrightarrow$ LexBFS* $\mathcal{O}(nm)$

Algorithm COSINE*

ALGORITHM COSINE (Hertz 1990)

Input: A graph G on n vertices and an ordering σ on its vertices.
Output: A coloring of the vertices of G.

Initialization: $c = 1$;
General step: While there exist uncolored vertices do:
1. While there exist uncolored vertices that have no neighbor colored c do:
   1.1. Let A be the set of uncolored vertices that have a neighbor colored c;
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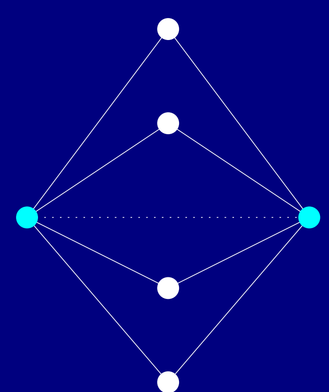

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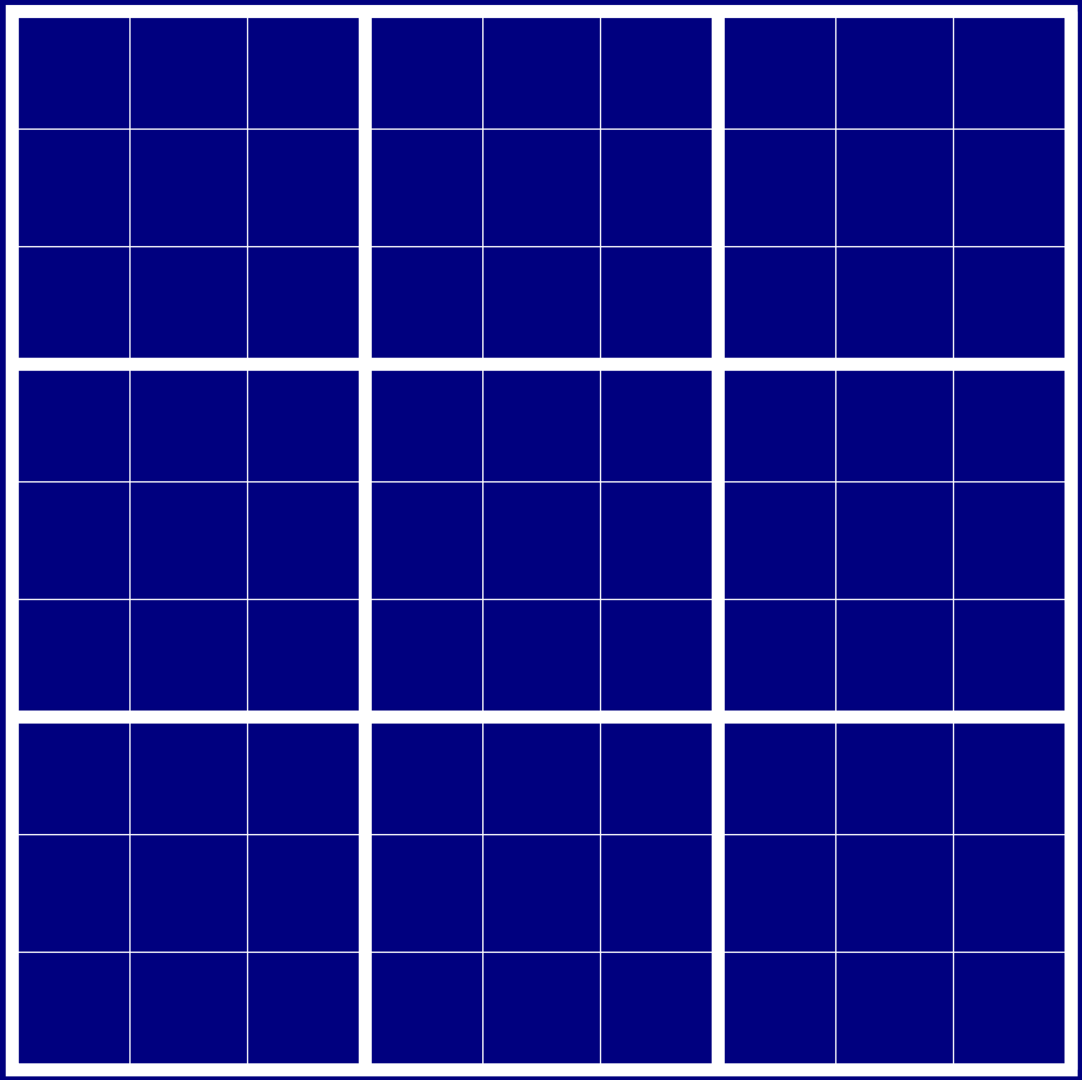
LEXBFS* on $\overline{G}$ + COSINE* on $G \rightarrow \mathcal{O}(nm)$ coloring algorithm

Precoloring Extension

Precoloring Extension

Sudoku



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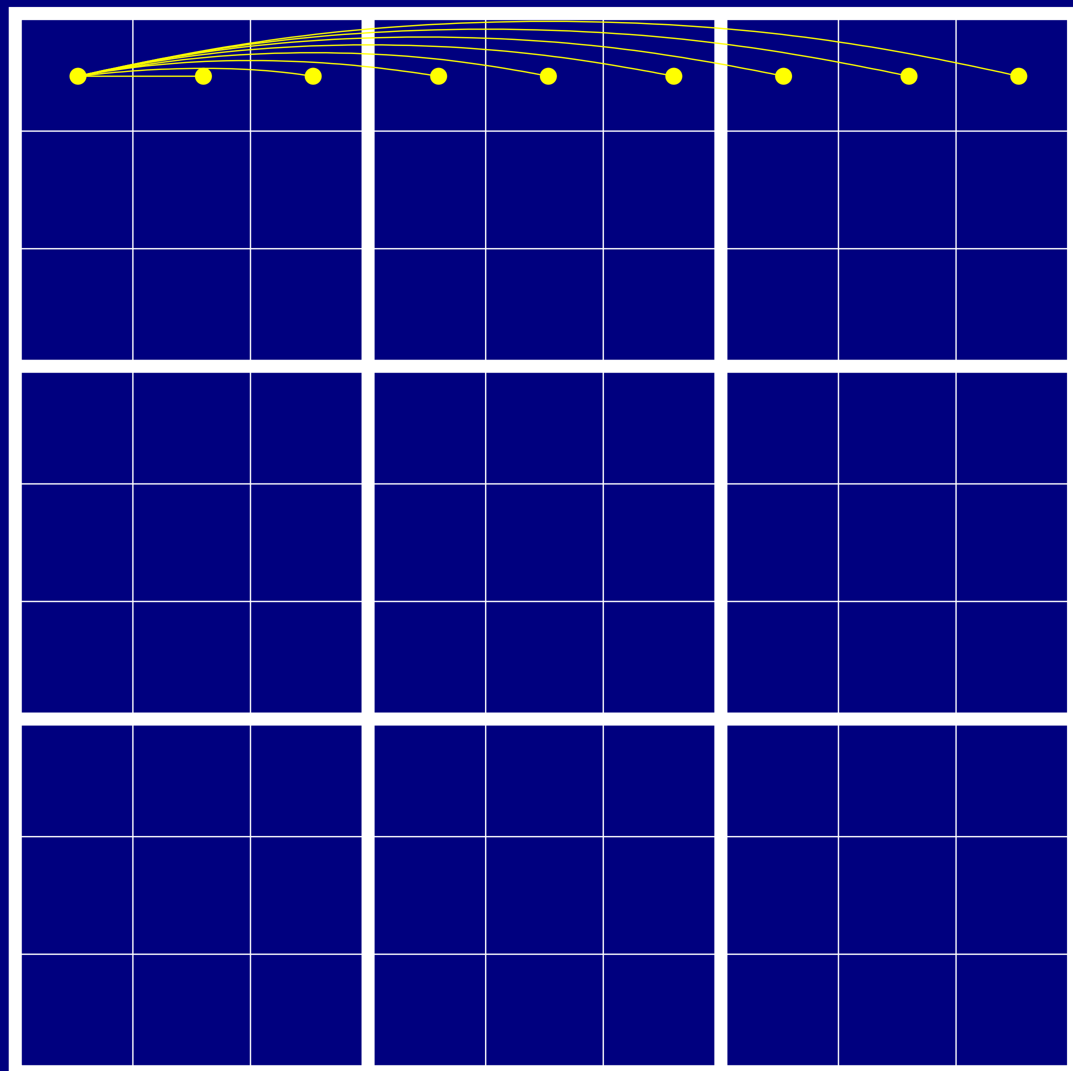

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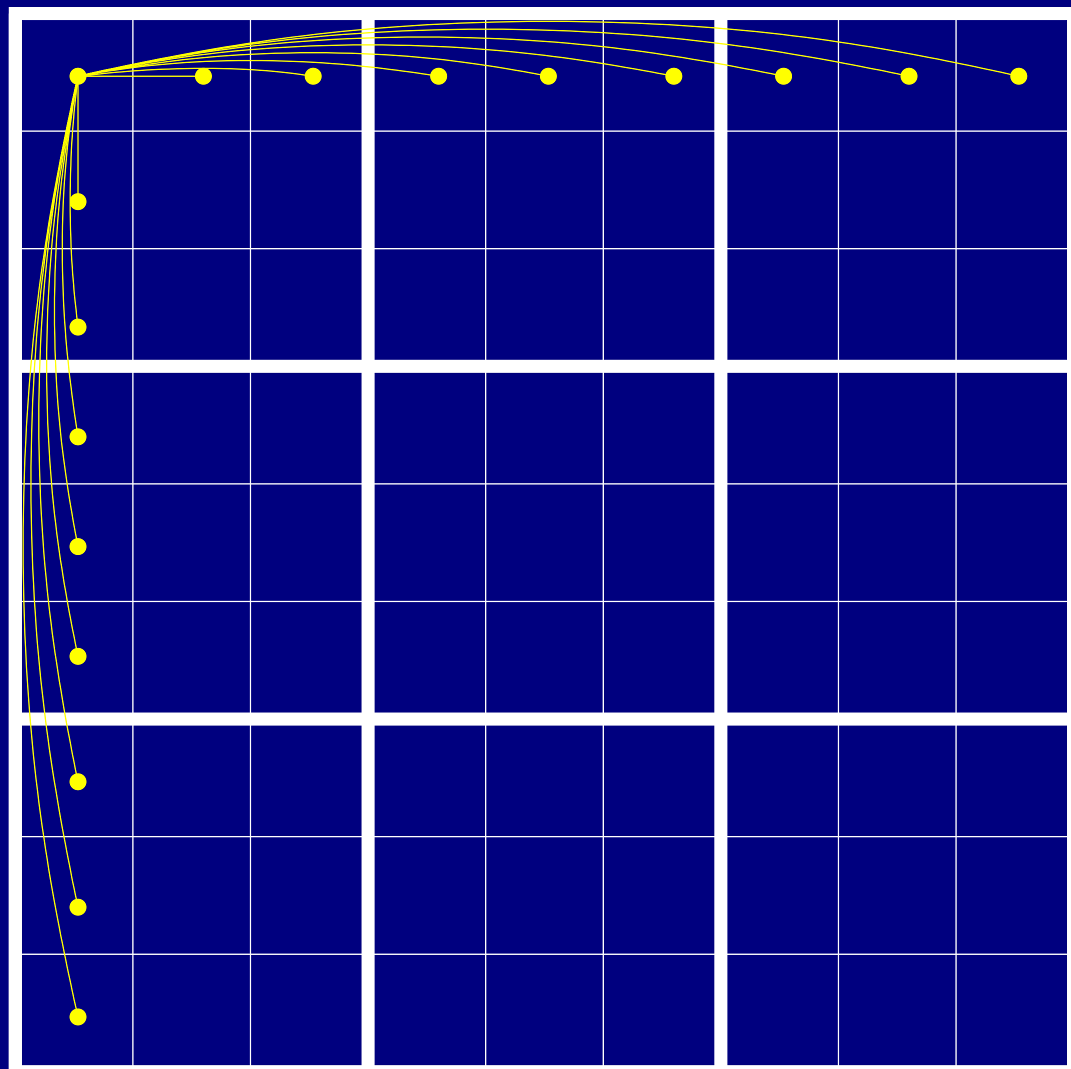

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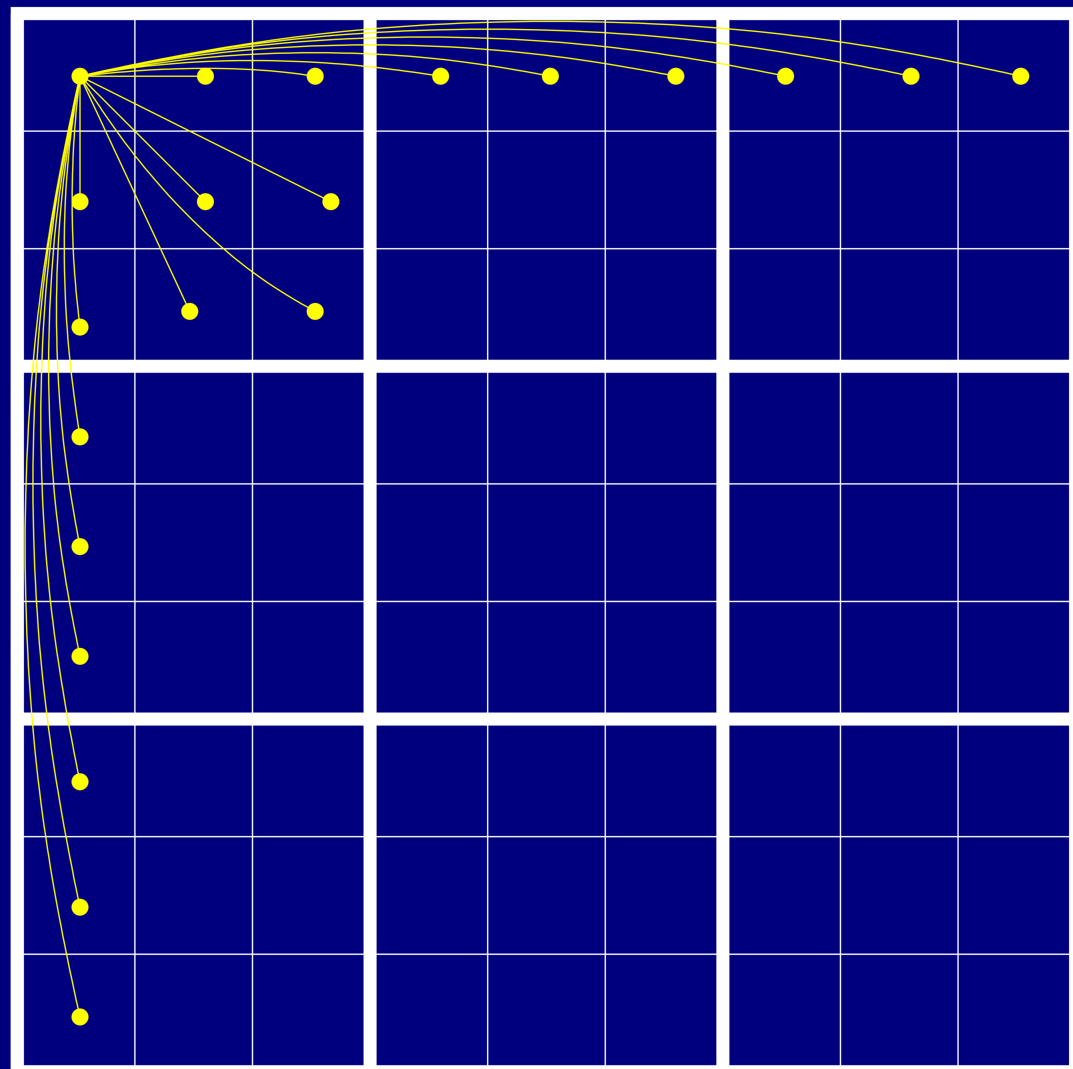

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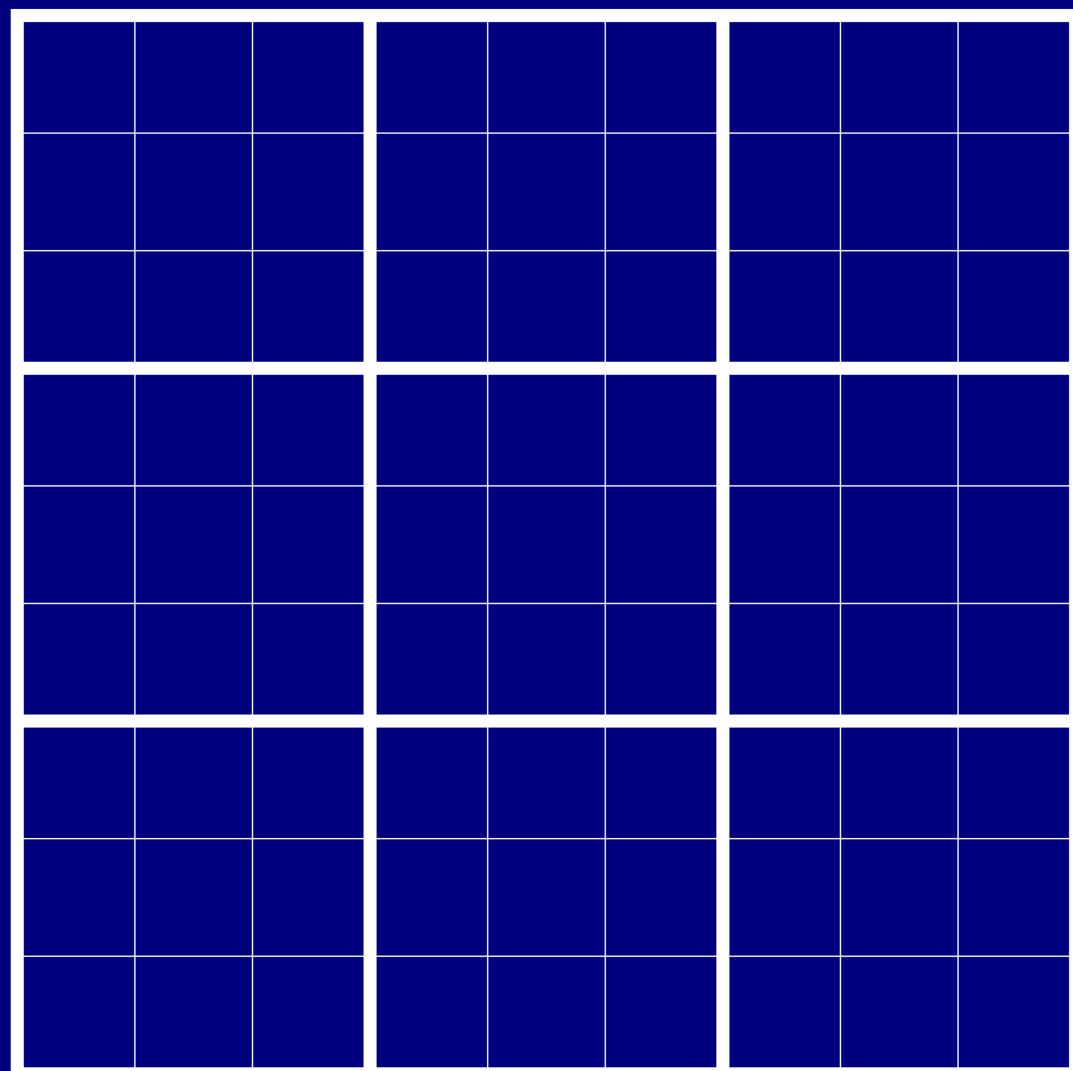


Precoloring Extension

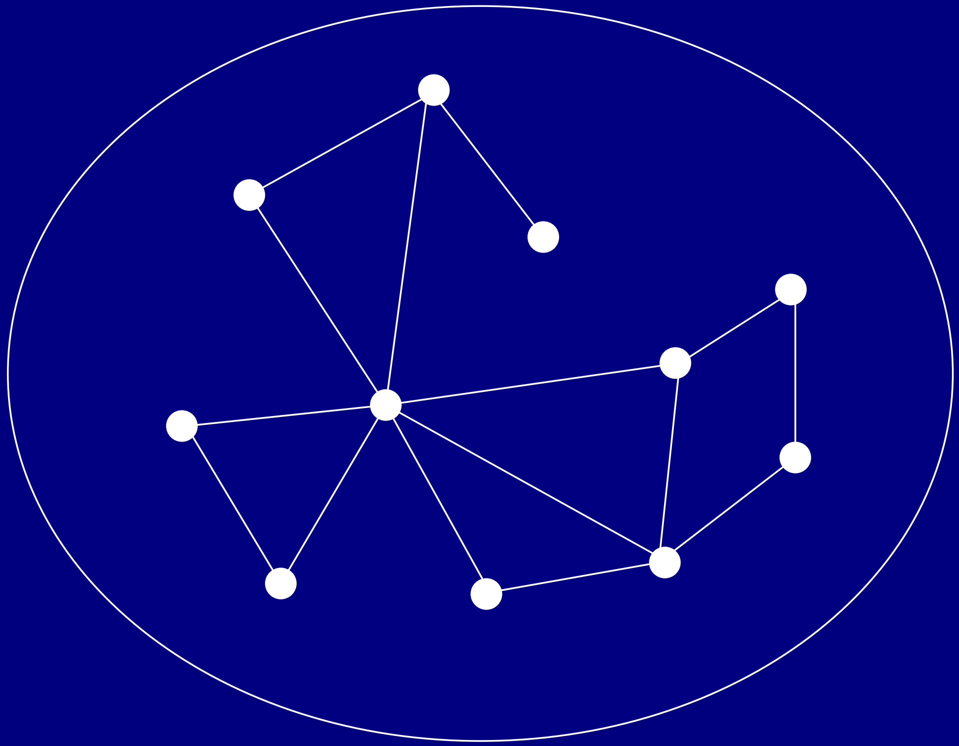
Sudoku

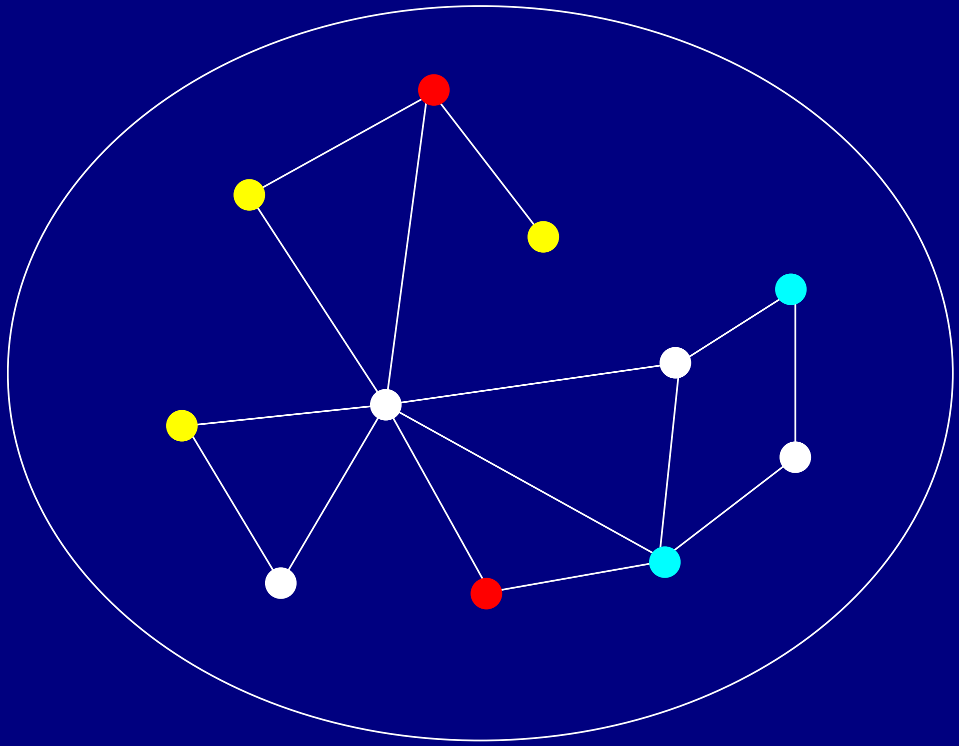
3	8		4		7		5	
				9			1	7
9	6	7		1				4
					8		2	9
6	9	2	5	7	4	3	8	1
7	3		1					
8				4		5	9	3
1	7			5				
	5		9		2		7	6

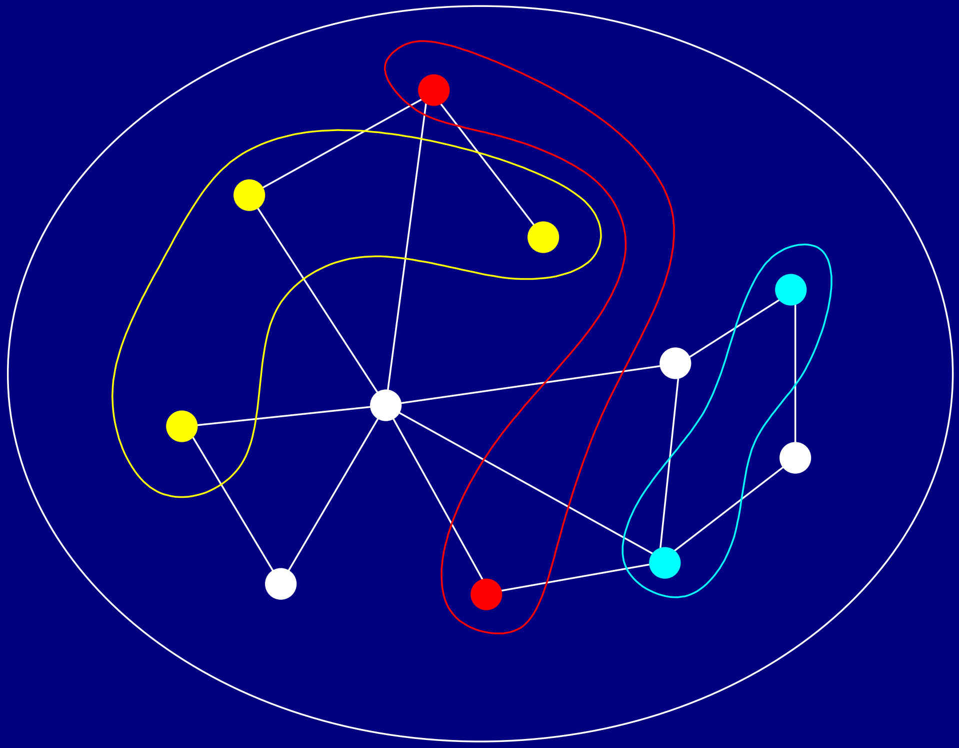
Precoloring Extension

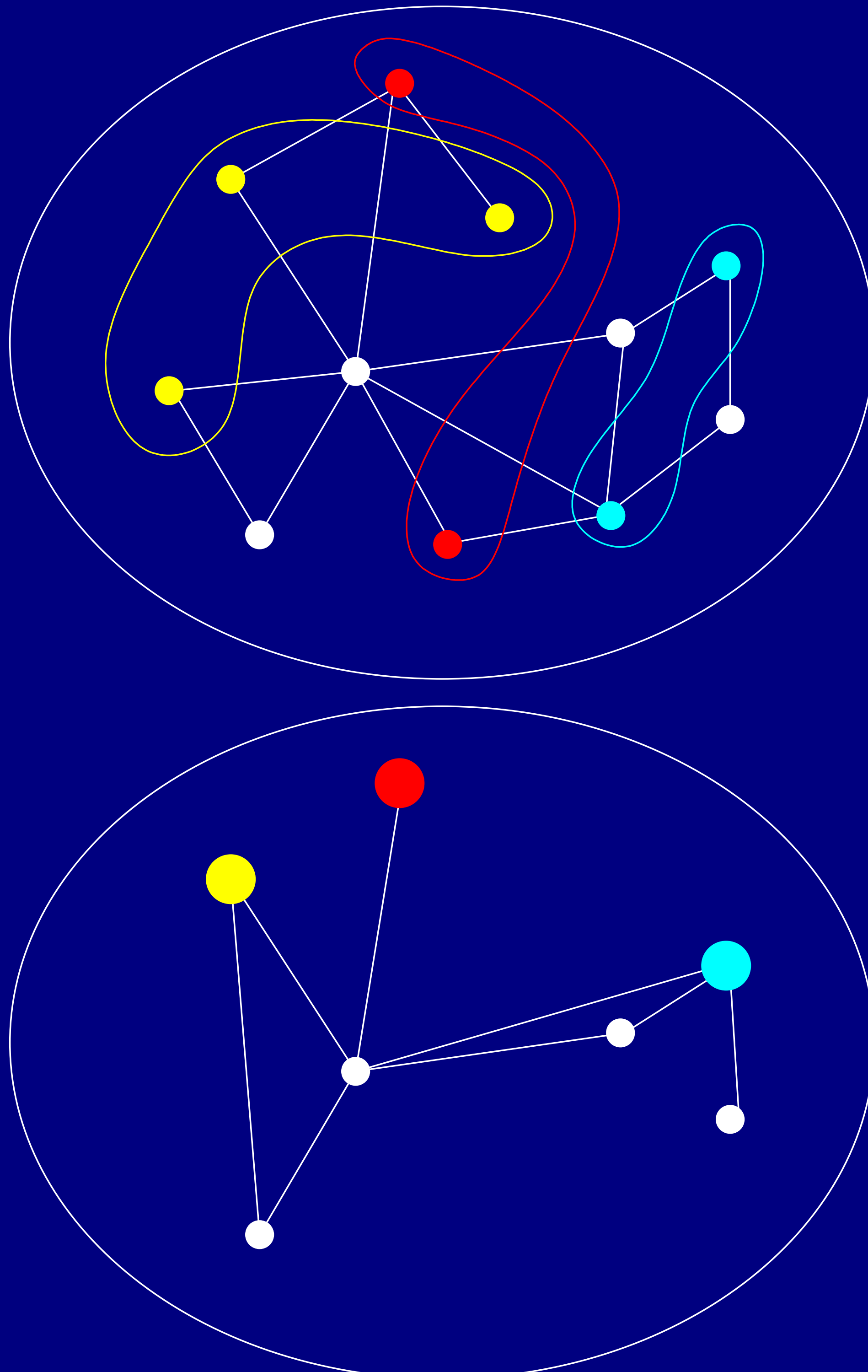
Sudoku

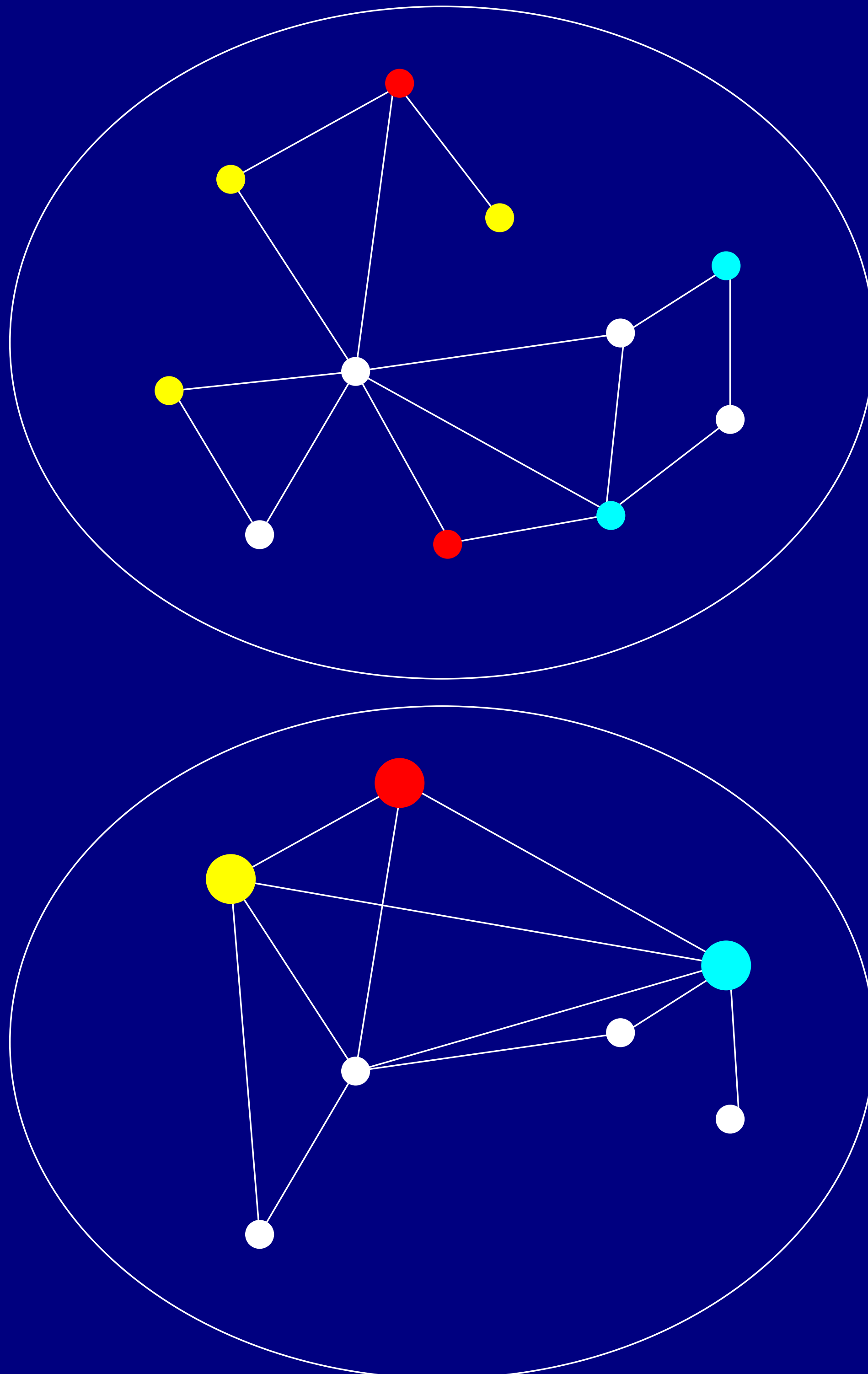
3	8	1	4	6	7	9	5	2
2	4	5	8	9	3	6	1	7
9	6	7	2	1	5	8	3	4
5	1	4	6	3	8	7	2	9
6	9	2	5	7	4	3	8	1
7	3	8	1	2	9	4	6	5
8	2	6	7	4	1	5	9	3
1	7	9	3	5	6	2	4	8
4	5	3	9	8	2	1	7	6

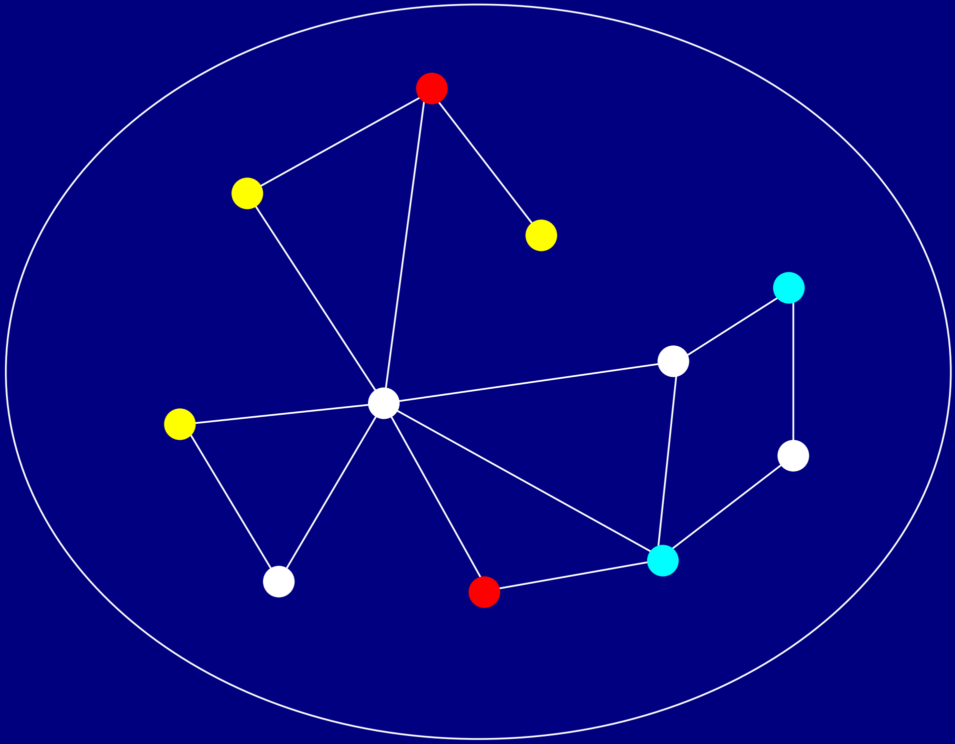


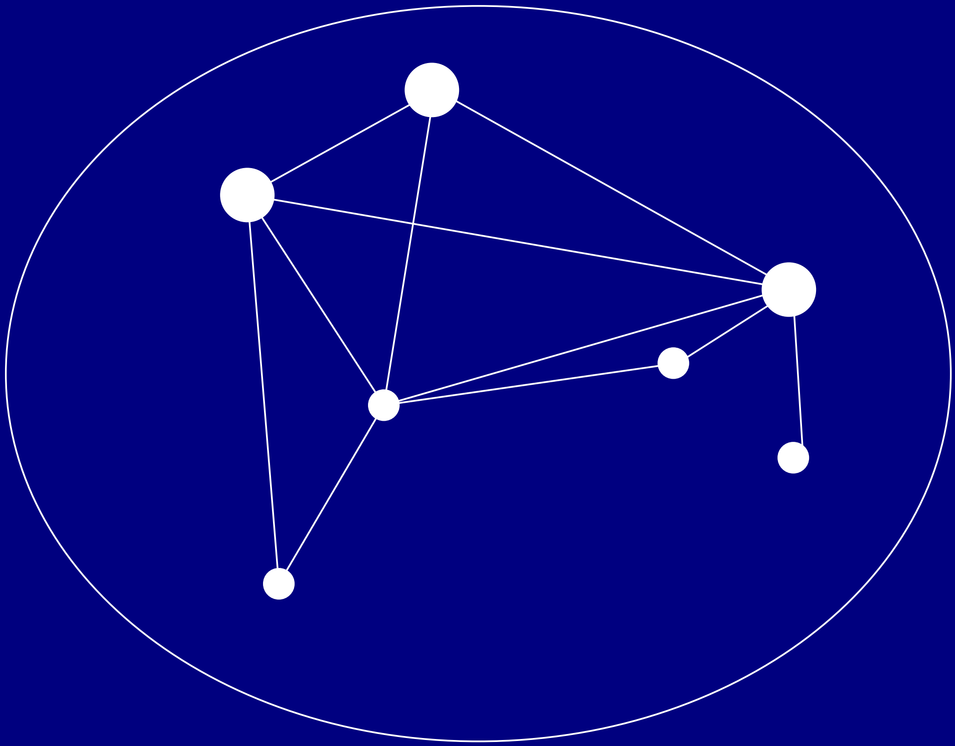












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PrExt-perfect graphs (Hujter, Tuza - 1996)

For every precoloring, the contracted graph is perfect

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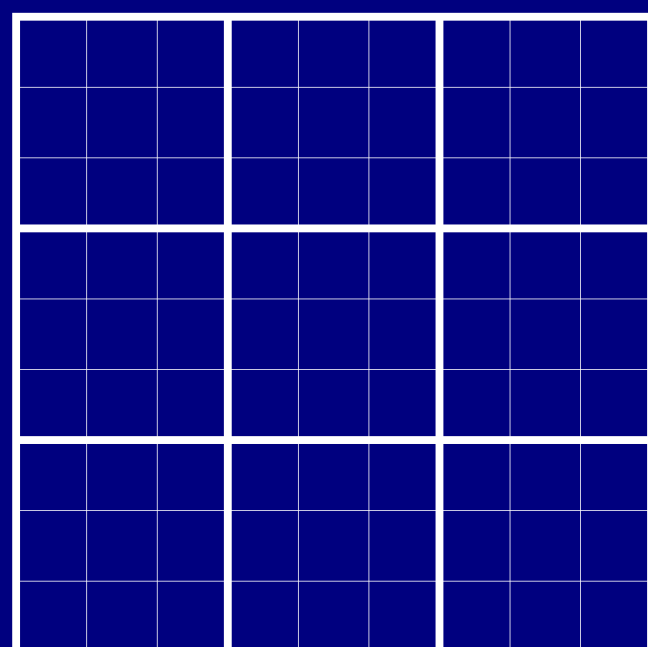

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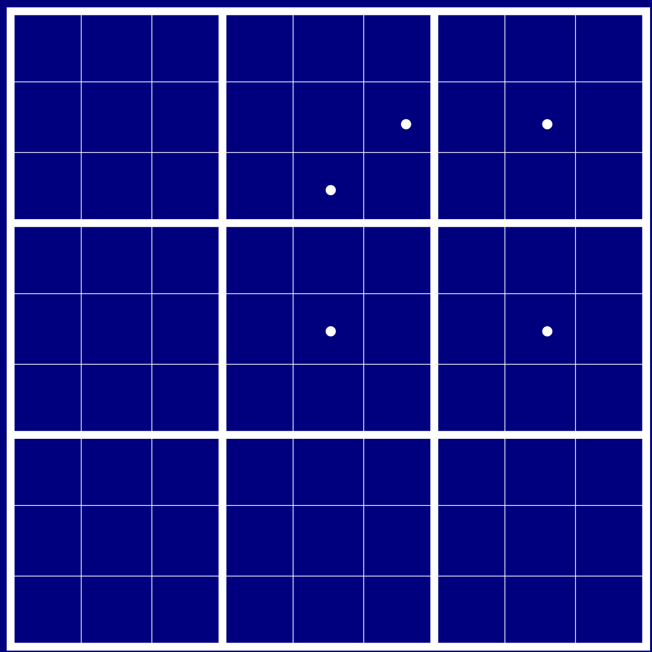

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