

## Some exercises on coherent lower previsions

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1. Assume we make the following judgements about the desirability of a few gambles:

	1	2	3	Desirable?
$f_1$	-1	0	1	Yes
$f_2$	0	2	-1	Yes
$f_3$	0	1	-1	No
$f_4$	1	-1	0	No

- (a) Are these assessments coherent?  
 (b) What do they imply about the desirability of the following gambles?:

	1	2	3
$f_5$	1	-1	-1
$f_6$	-2	4	1
$f_7$	1	0	0

- (c) What is their natural extension?  
 (d) What is the lower prevision of the following gamble?:

	1	2	3
$f_8$	2	3	1

2. Consider the lower prevision given by:

	$f(1)$	$f(2)$	$f(3)$	$\underline{P}(f)$
$f_1$	2	1	0	0.5
$f_2$	0	1	2	1
$f_3$	0	1	0	1

- (a) Does it avoid sure loss?

	1	2	3
$f_4$	2	1	0
$f_5$	-1	1	0

- (b) Is it coherent?  
(c) What is the natural extension on the gambles:

3. Let  $\underline{P}$  be the lower prevision on  $\mathcal{L}(\{1, \dots, 3\})$  given by

$$\underline{P}(f) = \frac{\min\{f(1), f(2), f(3)\}}{2} + \frac{\min\{f(1), f(2), f(3)\}}{2}.$$

Is it coherent?

4. **Vacuous lower previsions.** Let  $A$  be a non-empty subset of a (not necessarily finite) set  $\mathcal{X}$ . Say we only know that the lower probability of  $A$  is equal to 1. This assessment is embodied through the lower prevision  $\underline{P}$  defined on the singleton  $\{I_A\}$  by  $\underline{P}(I_A) = 1$  (again, recall that we denote  $\underline{P}(I_A)$  by  $\underline{P}(A)$ ).

- (a) Preparatory exercise. Show that the vacuous lower prevision relative to  $A$ , defined by

$$\underline{P}_A(f) := \inf_{x \in A} f(x) \tag{0.1}$$

for any  $f \in \mathcal{L}(\mathcal{X})$ , is a coherent lower prevision on  $\mathcal{L}(\mathcal{X})$ .

- (b) Show that  $\underline{P}$  avoids sure loss.  
(c) Show that  $\underline{P}$  is coherent.  
(d) Prove that the natural extension  $\underline{E}$  of  $\underline{P}$  is equal to the vacuous lower prevision relative to  $A$ :

$$\underline{E}(f) = \underline{P}_A(f) = \inf_{x \in A} f(x), \tag{0.2}$$

for any  $f \in \mathcal{L}(\mathcal{X})$ .

5. **Non-additive measures and the Choquet integral.** Suppose we have a 2-monotone set function  $\mu$  defined on the power set  $\wp(\mathcal{X})$  of a finite set  $\mathcal{X}$ , that is,

- (i)  $\mu(\emptyset) = 0$ ,  $\mu(\mathcal{X}) = 1$ ,  
(ii)  $\mu(A) \geq \mu(B)$  for all  $A, B \subseteq \mathcal{X}$  such that  $A \supseteq B$ , and  
(iii)  $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$  for any  $A, B \subseteq \mathcal{X}$ .

We may interpret the values  $\mu(A)$  as supremum buying prices for indicator gambles  $I_A$ . This corresponds to the lower prevision  $\underline{P}$  defined on  $\{I_A : A \subseteq \mathcal{X}\}$  by  $\underline{P}(I_A) := \mu(A)$  for all  $x \in \mathcal{X}$ .

The Choquet integral of a gamble  $f$  on a finite set  $\mathcal{X}$  can be constructed as follows. Since  $\mathcal{X}$  is finite, without loss of generality we can write  $f$  as

$$f = \alpha_0 + \sum_{i=1}^n \alpha_i I_{A_i} \quad (0.3)$$

with  $\alpha_0 \in \mathbb{R}$ ,  $\alpha_1 > 0$ ,  $\alpha_2 > 0$ ,  $\dots$ ,  $\alpha_n > 0$  and  $A_1 \supset A_2 \supset \dots \supset A_n$  (where  $A \supset B$  means  $A \supseteq B$  and  $A \neq B$ ). In terms of  $\alpha_i$ 's and  $A_i$ 's, the Choquet integral of  $f$  is simply given by

$$C \int f \, d\mu := \alpha_0 + \sum_{i=1}^n \alpha_i \mu(A_i). \quad (0.4)$$

- (a) Preparatory exercise. Show that the Choquet integral defines a coherent lower prevision. Use (i.e., do not prove) the sub-additivity theorem, which says that  $C \int (f + g) \, d\mu \geq C \int f \, d\mu + C \int g \, d\mu$  for all  $f, g \in \mathcal{L}(\mathcal{X})$ .
- (b) Show that the lower prevision  $\underline{P}$  representing  $\mu$  is coherent. [Hint: use (5a).]
- (c) Prove that the natural extension of  $\underline{P}$  coincides with the Choquet integral with respect to  $\mu$ .  
[Hint: show that  $C \int \cdot \, d\mu$  is the point-wise smallest coherent lower prevision on  $\mathcal{L}(\mathcal{X})$  which dominates  $\underline{P}$  on its domain.]

6. **Belief functions and random sets.** Suppose we have a probability measure  $\mu$  defined on the power set  $\wp(\mathcal{Y})$  of a finite set  $\mathcal{Y}$ , and a multi-valued mapping  $\Gamma$  from  $\mathcal{Y}$  into a finite set  $\mathcal{X}$ . As Dempster (1967) puts it: “if the uncertain outcome  $y$  is known to correspond to an uncertain outcome  $x \in \Gamma(y)$ , what probability judgements may be made about the uncertain outcome  $x \in \mathcal{X}$ ?” For the sake of simplicity, we shall assume that  $\Gamma(y) \neq \emptyset$  for all  $y \in \mathcal{Y}$ .

Consider the set function  $\nu$  on  $\wp(\mathcal{X})$  defined as

$$\nu(A) := \mu(\{y \in \mathcal{Y} : \Gamma(y) \subseteq A\}) \quad (0.5)$$

for every  $A \subseteq \mathcal{X}$ .

We may interpret the values  $\nu(A)$  as supremum buying prices for indicator gambles  $I_A$ . This corresponds to the lower prevision  $\underline{P}$  defined on  $\{I_A : A \subseteq \mathcal{X}\}$  by  $\underline{P}(I_A) := \nu(A)$  for all  $A \subseteq \mathcal{X}$ .

- (a) Preparatory exercise. Show that

$$\underline{P}(I_A) = \sum_{y \in \mathcal{Y}} \mu(\{y\}) \underline{P}_{\Gamma(y)}(I_A). \quad (0.6)$$

- (b) Prove that  $\nu$  is a 2-monotone set function, as defined in Exercise 5. [Hint: first show that for all  $y \in \mathcal{Y}$ ,  $\underline{P}_{\Gamma(y)}$  is 2-monotone as a set function restricted to events, and then use (6a).]
- (c) Show that  $\underline{P}$  is coherent. [Hint: either rely on results derived previously in Exercise 4, or rely on Exercise 5.]
- (d) Show that the natural extension of  $\underline{P}$  is given by

$$\underline{E}(f) = \sum_{y \in \mathcal{Y}} \mu(\{y\}) \underline{P}_{\Gamma(y)}(f). \quad (0.7)$$

- (e) Prove that  $\underline{E}$  is the  $\mathcal{X}$ -marginal of the marginal extension of  $P_\mu$  and  $\underline{P}_\Gamma(\cdot|\mathcal{Y})$ , where

$$P_\mu(f) := \int f \, d\mu = \sum_{y \in \mathcal{Y}} \mu(\{y\}) f(y) \quad (0.8)$$

for all  $f \in \mathcal{L}(\mathcal{Y})$ , and

$$\underline{P}_\Gamma(f|y) := \underline{P}_{\Gamma(y)}(f) \quad (0.9)$$

for all  $f \in \mathcal{L}(\mathcal{X})$  and  $y \in \mathcal{Y}$ . Hence,  $\nu$  is indeed the (least committal) lower probability following from the premises.

- (f) Show that, for all gambles  $f$  on  $\mathcal{X}$ ,

$$\mathbb{C} \int f \, d\nu = \sum_{y \in \mathcal{Y}} \mu(\{y\}) \underline{P}_{\Gamma(y)}(f). \quad (0.10)$$

[Hint: use (6d), (6b), and a result from Exercise 5.]

7. **Possibility and necessity measures.** Suppose we have a minitive set function  $\nu$  defined on the power set  $\wp(\mathcal{X})$  of a finite set  $\mathcal{X}$ , that is,

- (i)  $\nu(\emptyset) = 0$ ,  $\nu(\mathcal{X}) = 1$ ,
- (ii)  $\nu(A) \geq 0$  for all  $A \subseteq \mathcal{X}$ , and
- (iii)  $\nu(A \cap B) = \min\{\nu(A), \nu(B)\}$  for any  $A, B \subseteq \mathcal{X}$ .

We may interpret the values  $\nu(A)$  as supremum buying prices for indicator gambles  $I_A$ . This corresponds to the lower prevision  $\underline{P}$  defined on  $\{I_A : A \subseteq \mathcal{X}\}$  by  $\underline{P}(I_A) := \nu(A)$  for all  $x \in \mathcal{X}$ .

- (a) Show that  $\nu$  is a necessity measure.
- (b) Show that, for every  $A \subseteq \mathcal{X}$ ,  $A \neq \mathcal{X}$ ,

$$\nu(A) = \min_{x \in A^c} \nu(\{x\}^c) \quad (0.11)$$

- (c) Show that  $\nu$  is 2-monotone.

- (d) Show that  $\nu$  is a belief function. Start with defining  $n(x) := \nu(\{x\}^c)$ . Let  $y_1, \dots, y_m$  be an enumeration of the values of  $n$  with  $y_1 < y_2 < \dots < y_m$  (note that  $y_1 = 0$  because  $\nu(\emptyset) = 0$ ). Now let  $\mathcal{Y} = \{y_1, \dots, y_m\}$  and define the multi-valued mapping

$$\Gamma(y_i) := A_i \text{ where } A_i := \{x \in \mathcal{X} : n(x) \leq y_i\}. \quad (0.12)$$

Find a probability measure  $\mu$  on  $\mathcal{Y}$  such that

$$\nu(A) = \min_{x \in A^c} n(x) = \sum_{i=1}^n \mu(\{y_i\}) \underline{P}_{\Gamma(y_i)}(I_A). \quad (0.13)$$

[Hint: First show that  $\nu(A_i) = y_{i+1}$  for all  $i < m$ , and continue from there.]

8. **P-boxes.** Let  $\mathcal{X} = \mathbb{R}$ . Let  $x_1, x_2 \in \mathbb{R}$ ,  $x_1 < x_2$ . Consider the linear previsions  $\underline{P}_{x_1}$  and  $\underline{P}_{x_2}$  defined by

$$\underline{P}_{x_1}(f) := f(x_1), \quad (0.14)$$

$$\underline{P}_{x_2}(f) := f(x_2), \quad (0.15)$$

for all  $f \in \mathcal{L}(\mathcal{X})$ . Note that these linear previsions are vacuous lower previsions relative to singletons. The lower envelope  $\underline{P}$  of  $\underline{P}_{x_1}$  and  $\underline{P}_{x_2}$  is nothing but the vacuous lower prevision relative to the pair  $\{x_1, x_2\}$ :

$$\underline{P}(f) = \min\{f(x_1), f(x_2)\}. \quad (0.16)$$

Note that  $\underline{P}$  is coherent.

- (a) Draw the p-box that corresponds to  $\underline{P}$ .  
 (b) Prove that the “natural extension” of this p-box, that is, the lower envelope  $\underline{E}$  of all linear previsions  $Q \in \mathcal{P}(\mathcal{X})$  whose cumulative distribution function

$$F_Q(x) = Q(\{y \in \mathcal{X} : y \leq x\}) \quad (0.17)$$

belongs to this p-box, is dominated by the vacuous lower prevision relative to the *interval*  $[x_1, x_2]$ , that is,

$$\underline{E}(f) \leq \underline{P}_{[x_1, x_2]}(f) \text{ for any gamble } f \in \mathcal{L}(\mathcal{X}). \quad (0.18)$$

What does this mean?

- (c) Extra exercise. If you are fond of  $\epsilon$ 's, show that

$$\underline{E}(f) = \sup_{\epsilon > 0} \underline{P}_{(x_1 - \epsilon, x_2]}(f) \text{ for any gamble } f \in \mathcal{L}(\mathcal{X}). \quad (0.19)$$

9. Consider two binary random variables  $X_1, X_2$ , and let  $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$  be given by:

$$\begin{aligned}\underline{P}(f|X_2 = 0) &= \min \left\{ \frac{f(0,0) + f(1,0)}{2}, f(0,0) \right\} \\ \underline{P}(f|X_2 = 1) &= \min \left\{ \frac{f(0,1) + f(1,1)}{2}, f(1,1) \right\} \\ \underline{P}(f|X_1 = 0) &= \min \left\{ \frac{f(0,0) + f(0,1)}{2}, f(0,0) \right\} \\ \underline{P}(f|X_1 = 1) &= \min \left\{ \frac{f(1,0) + f(1,1)}{2}, f(1,1) \right\}\end{aligned}$$

for any gamble  $f$  on  $\{0,1\}^2$ . Are these conditional lower previsions coherent?

10. Consider  $\mathcal{X}_1 = \mathcal{X}_2 = \{1,2,3\}$ , and let  $\mathcal{M}$  be the set of probability mass functions on  $\mathcal{X}_1 \times \mathcal{X}_2$  satisfying  $P(1,2) = P(2,2) = P(3,1) = 0, P(1,1) = P(2,1), P(1,1) \geq P(1,3), P(2,1) \leq P(2,3)$ , where the first index denotes the value of  $X_1$  and the second the value of  $X_2$ . Let  $\underline{P}$  be the lower envelope of the set  $\mathcal{M}$ .
- Compute the regular extensions  $\underline{R}(X_1|X_2), \underline{R}(X_2|X_1)$ .
  - Compute the natural extensions  $\underline{E}(X_1|X_2), \underline{E}(X_2|X_1)$ .
  - Define  $\underline{P}(X_2|X_1)$  from  $\underline{P}$  using regular extension, and let  $\underline{P}(X_1|X_2 = x)$  be defined from  $\underline{P}$  by natural extension if  $x = 3$  and by regular extension otherwise. Are  $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$  weakly coherent with  $\underline{P}$ ?
  - Are they coherent?
11. **The two envelopes problem.** The aim of this exercise is to demonstrate how mixing can annihilate imprecision, and how extra information does not necessarily lead to extra precision, when updating using Bayes rule. This latter phenomenon is called *dilation*.

I have two sealed envelopes, both containing money. One of them contains twice as much money as the other. You are free to pick one of them. You open it and find 100 Euro inside. You are provided the choice of either keeping the 100 Euro, or switching with whatever amount there is in the other envelope, which you know to be either 50 Euro or 200 Euro. Should you switch or not?

Let's introduce a few random variables. Let  $X$  be the amount in your envelope. Let  $Y$  be the smallest amount in the envelopes. Let  $Z$  be 1 if your envelope has the lowest value and 2 if the your envelope has the highest value. So,

$$X = YZ. \tag{0.20}$$

Since you pick at random, you know that the probability of  $Z = 1$  is  $\frac{1}{2}$ , as is the probability of  $Z = 2$ . Moreover, since your choice is independent of how the money was distributed in the envelopes,  $Y$  is irrelevant to  $Z$ . A priori, we know nothing about  $Y$ .

It is evident that once  $X = 100$  has been observed, if  $Z = 1$  then  $Y = 100$  and we should switch, and if  $Z = 2$  then  $Y = 50$  and we should not switch. However, we do not know  $Z$ ; we only know that  $Z$  is uniformly distributed over  $\{1, 2\}$ . What should we do?

- (a) Let  $f$  be a gamble on  $\mathcal{Z}$  (with  $\mathcal{Z} = \{1, 2\}$ ). For any  $y \in \mathcal{Y}$  (with  $\mathcal{Y} = \mathbb{R}^+$ ), what is the conditional lower prevision  $\underline{P}(f|Y = y)$ ?
- (b) Let  $f$  be a gamble on  $\mathcal{Y}$ . What is the marginal lower prevision  $\underline{P}(f)$ ?
- (c) Let  $f$  be a gamble on  $\mathcal{Y} \times \mathcal{Z}$ . Use (11a), (11b), and marginal extension, to arrive at an expression for  $\underline{P}(f)$ .
- (d) Calculate  $\mathcal{M}(\underline{P})$ : show that  $Q \in \mathcal{M}(\underline{P})$  if and only if there is a linear prevision  $R$  on  $\mathcal{L}(\mathcal{Y})$  such that  $Q(f) = \frac{1}{2}R(f(\cdot, 1) + f(\cdot, 2))$  for all gambles  $f$  on  $\mathcal{Y} \times \mathcal{Z}$ . [Hint: use (11c), and the fact that there is a one-to-one correspondence between convex and compact sets of linear previsions and coherent lower previsions. (You do not need to prove compactness.)]
- (e) Conclude that  $Q \in \mathcal{M}(\underline{P})$  if and only if  $Q(f) = \frac{1}{2}Q(f(\cdot, 1) + f(\cdot, 2))$  for all gambles  $f$  on  $\mathcal{Y} \times \mathcal{Z}$ .
- (f) What are the *prior* lower and upper previsions—before you open the envelope that you picked—of the amount in the other envelope minus the amount in your envelope? [Hint: consider every linear prevision  $Q$  in  $\mathcal{M}(\underline{P})$  and invoke (11e).]
- (g) What can you say about the posterior distribution—after observing  $X = 100$ —of  $Z$ ? That is, for every  $Q$  in  $\mathcal{M}(\underline{P})$  such that  $Q(X = 100) > 0$ , what are  $Q(Z = 1|X = 100)$  and  $Q(Z = 2|X = 100)$ ? (Note that the envelope of these probabilities is called *regular extension*.) Express these probabilities in terms of the distribution of  $Y$  under  $Q$  conditional on the event  $(Y = 100 \text{ or } Y = 50)$ .
- (h) What can you say about the posterior prevision—after observing  $X = 100$ —of the amount in the other envelope minus the amount in your envelope? Again consider every  $Q$  in  $\mathcal{M}(\underline{P})$  such that  $Q(X = 100) > 0$ .

12. **The three prisoners problem.** Three men,  $a$ ,  $b$  and  $c$ , are in jail. Prisoner  $a$  knows that only two of the three prisoners will be executed, but he doesn't know who will be spared. He only knows that all three prisoners have equal probability  $\frac{1}{3}$  of being spared. To the warden who knows which prisoner will be spared,  $a$  says, "Since two out of the three will be executed, it is certain that either  $b$  or  $c$  will be. You will give me no information about my own chances if you give me the name of one man,

$b$  or  $c$ , who is going to be executed.” Accepting this argument after some thinking, the warden says, “Prisoner  $b$  will be executed.”

Does the warden’s statement truly provide no information about the chance of  $a$  to be executed? We try to solve this problem using the theory of lower previsions.

- (a) Let the variable  $X$  denote the prisoner that will be spared. Since all three prisoners have equal probability  $\frac{1}{3}$  of being spared, we have a prior prevision specified by  $\underline{P}_0(\{a\}) = \underline{P}_0(\{b\}) = \underline{P}_0(\{c\}) = \overline{P}_0(\{a\}) = \overline{P}_0(\{b\}) = \overline{P}_0(\{c\}) = \frac{1}{3}$ . In a previous exercise, we have shown that the natural extension of  $\underline{P}_0$  is given by

$$\underline{E}_0(f) = \frac{1}{3}(f(a) + f(b) + f(c)). \quad (0.21)$$

for any  $f \in \mathcal{L}(\mathcal{X})$ .

- (b) Let the variable  $Y$  denote the prisoner named by the warden. Since the warden will not name  $a$ , we know that if  $X = a$ , then  $Y$  will be  $b$  or  $c$ , if  $X = b$  then  $Y = c$  and if  $X = c$  then  $Y = b$ . Such information is modelled by vacuous conditional lower previsions, again, as described in one of the previous exercises:

$$\underline{P}(g|X = a) = \min\{g(b), g(c)\} \quad (0.22)$$

$$\underline{P}(g|X = b) = g(c) \quad (0.23)$$

$$\underline{P}(g|X = c) = g(b) \quad (0.24)$$

for any gamble  $g \in \mathcal{L}(\mathcal{Y})$ . Note that in case  $X = a$ , we do not know the mechanism by which the warden names either  $b$  or  $c$  for  $Y$ . Therefore, it seems appropriate to model this situation through a vacuous lower prevision relative to  $\{b, c\}$ .

- (c) Combine the lower previsions  $\underline{E}_0(\cdot)$  on  $\mathcal{L}(\mathcal{X})$  and  $\underline{P}(\cdot|X)$  on  $\mathcal{L}(\mathcal{Y})$ , using the marginal extension theorem, to a coherent lower prevision  $\underline{E}$  on  $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ .
- (d) Apply the generalised Bayes rule to calculate  $\underline{E}(X = a|Y = b)$ ,  $\overline{E}(X = a|Y = b)$  and  $\underline{E}(X \neq a|Y = b)$ ,  $\overline{E}(X \neq a|Y = b)$ .
- (e) Extra exercise. After naming prisoner  $b$  as one of the prisoners to be executed, the warden thinks a little more and decides to play the following slightly sadistic game with prisoner  $a$ . The warden continues: “Are you really sure that I have given you no information at all by naming  $b$ ? If you want to, for a reasonable fee I can arrange your fate to be switched with the fate of prisoner  $c$ . Of course, since I have not given you any information at all, you might not care about such arrangement. On the other hand, switching with prisoner  $c$  might just save your life. . . It’s up to you to decide!”

Assume the utility of your life is equal to 25,000,000 Cuban Peso and the bribe requested by the warden is 25,000 Cuban Peso. Assuming



that the warden really tells the truth about being able to arrange the switch, what would you do if you were prisoner  $a$ ? (If the value of the bribe is zero, this game is isomorphic to the Monty Hall puzzle, as for instance described in de Cooman & Zaffalon, “Updating beliefs with incomplete observations”, *Artificial Intelligence*, 2004, 159, pp.75-125.)