PROOF NETS AND COHERENCE SPACES (oldies)

Talk @ **THOMAS EHRHARD 60** Festschrift

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HISTORICAL REMARKS

- Old stuff for the birthday
  of an old friend
  — who still is not that old!
- Quite an easy talk
  just a warm up with low level reminders
  before serious talks take place
- Thomas from 1986 – 1987 DEA Girard / Krivine
- Friends and neighbours with Pasquale Malacaria
  around rue mouffetard 1991
- Visits Marseilles / Nice in the early 90s
  (highly coherent period)
- Then workshop, committees etc.

Trade unionist remarks

- About this 29 september 2022 in France
- Secondary school teachers, school teachers are on strike
- They have been loosing a lot of purchasing power in the last 20 years
- Worse their working conditions have deteriorated considerably since the 80s
- Not enough persons want to become teachers anymore.

Without teachers we would not be academics today

Coherence spaces (Girard 1986)

A short reminder – That's where linear logic took place, well worth a visit!

Denotational semantics, categorical interpretation

Proof π of C under asumption H

Morphism from [[H]] to [[C]]

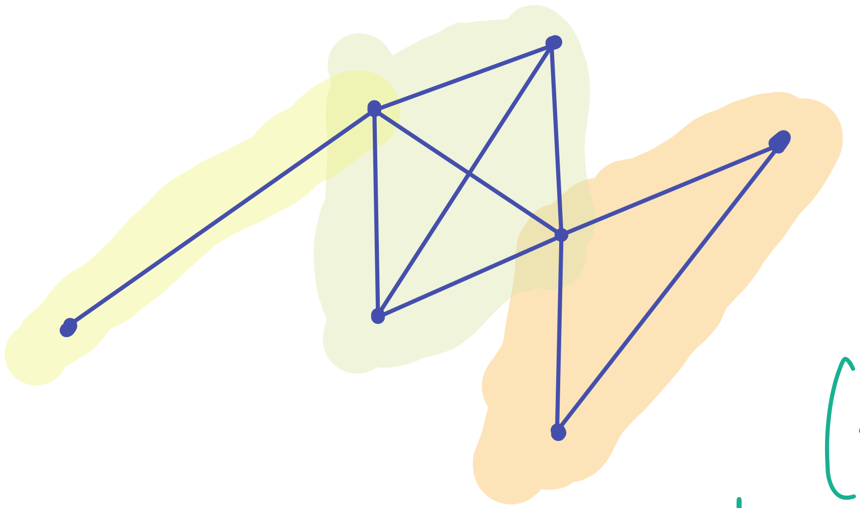
When π reduces to π' … π unchanged i.e. [[π]]=[[π']]

(full abstraction, denotational completeness....
Sequentiality ;-)

Coherence space A

Web : (countable set of tokens) |A|

A binary irreflexive relation on it $\frown$ (simple graph)



objects:
cliques
(not all of them)
maximal,
total ...

Linear Negation: complement graph
$a \frown b\ [A]$ iff $a \smile b\ [A^\perp]$

Stable maps

stable $\beta \in F(a)$

$\exists\, a_0 \subset A$ min $\beta \in F(a_0)$

$F : (a_0, \beta)$

Approximants

Representation of Hom(A,B) as a coherence space.

F : cliques to cliques

- $a \subset b \quad F(a) \subset F(b)$
- $F(\bigcup a_i) = \bigcup (F(a_i))$ $\quad \bigcup$: directed union
- if $a \cup b$ clique then $F(a \cap b) = F(a) \cap F(b)$

with & product $\quad$ C C C

Linear maps

Coherence spaces with different morphisms.

$\beta \in F(x)$
$\exists \alpha \in A \mid \beta \in F(\alpha)$
$F : (\alpha, \beta)$

linear maps instead of stable maps

Not a CCC $\otimes$ is not a product.

$F(\bigcup a_i) = \bigcup F(a_i)$ when all a_i a_j are pairwise compatible

(or F definable on tokens)

Linear connectives

$|A \ast B| = |A| \times |B|$

just the two of them (when commutative)

b, b'

$\otimes$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\smile$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\smile$	$\frown$	$\frown$

$a\ a'$

$⅋$	$\smile$	$=$	$\frown$
$\smile$	$\smile$	$\smile$	$\frown$
$=$	$\smile$	$=$	$\frown$
$\frown$	$\frown$	$\frown$	$\frown$

$(a,b) \overset{?}{\frown} (a',b')$

$|!A|$ = finite cliques of $|A|$ $\quad a \frown a'$ $\quad$ $C[!A]$ iff $a \cup b$ clique of A

$F(a) = \{\beta \mid \alpha \in a$ et $(\alpha, \beta) \in f\}$

$A \multimap B = A^\perp ⅋ B$

$A \Rightarrow B = (!A) \multimap B$

(Multiplicative) Linear Logic (Girard 1987)

Another short reminder

Axioms, Rules and Cuts (one sided multiplicative)

$$\vdash a, a^\perp \quad (a \vee \neg a)$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\otimes$$

contexts are concatenated

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A ⅋ B, \Delta}⅋$$

Cut

$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

(kind of $\otimes$ $\exists x \; x \otimes x^\perp$)

+ exchange (or not??)

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Remarks

Extremely simple calculus

→ very elegant, with many properties

→ not very expressive

The MIX rule

"and" implies "or" in MLL

$$\dfrac{\vdash \Gamma \qquad\qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ \text{mix}$$

$A \otimes B \multimap A \parr B$ validated by coherence spaces.
(coherent wrt $\otimes$ implies coherent wrt $\parr$)

with units ... complicated.

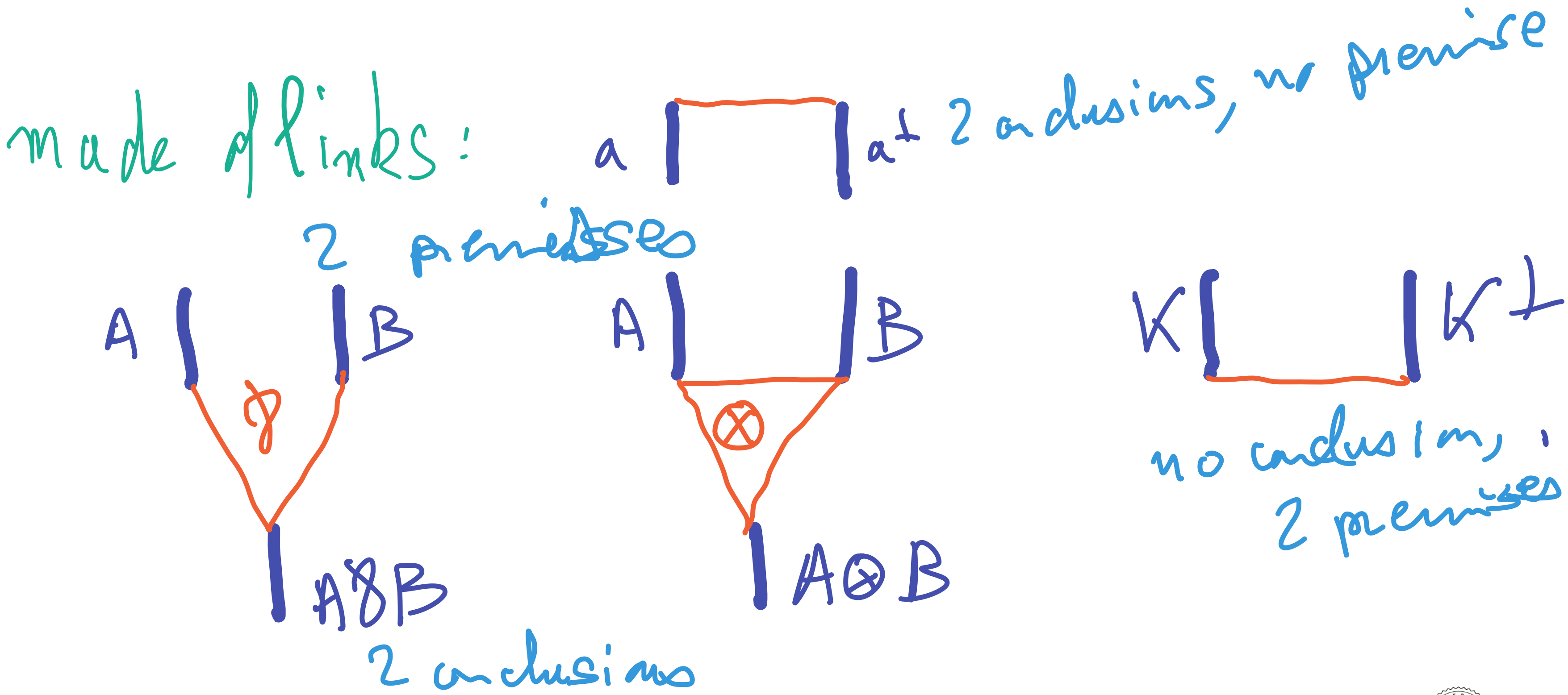
Multiplicative Proof nets (Girard 1987)

Yet another short reminder

Graphs denoting proofs

Up to rule permutations

Cut-elimination as graph rewriting



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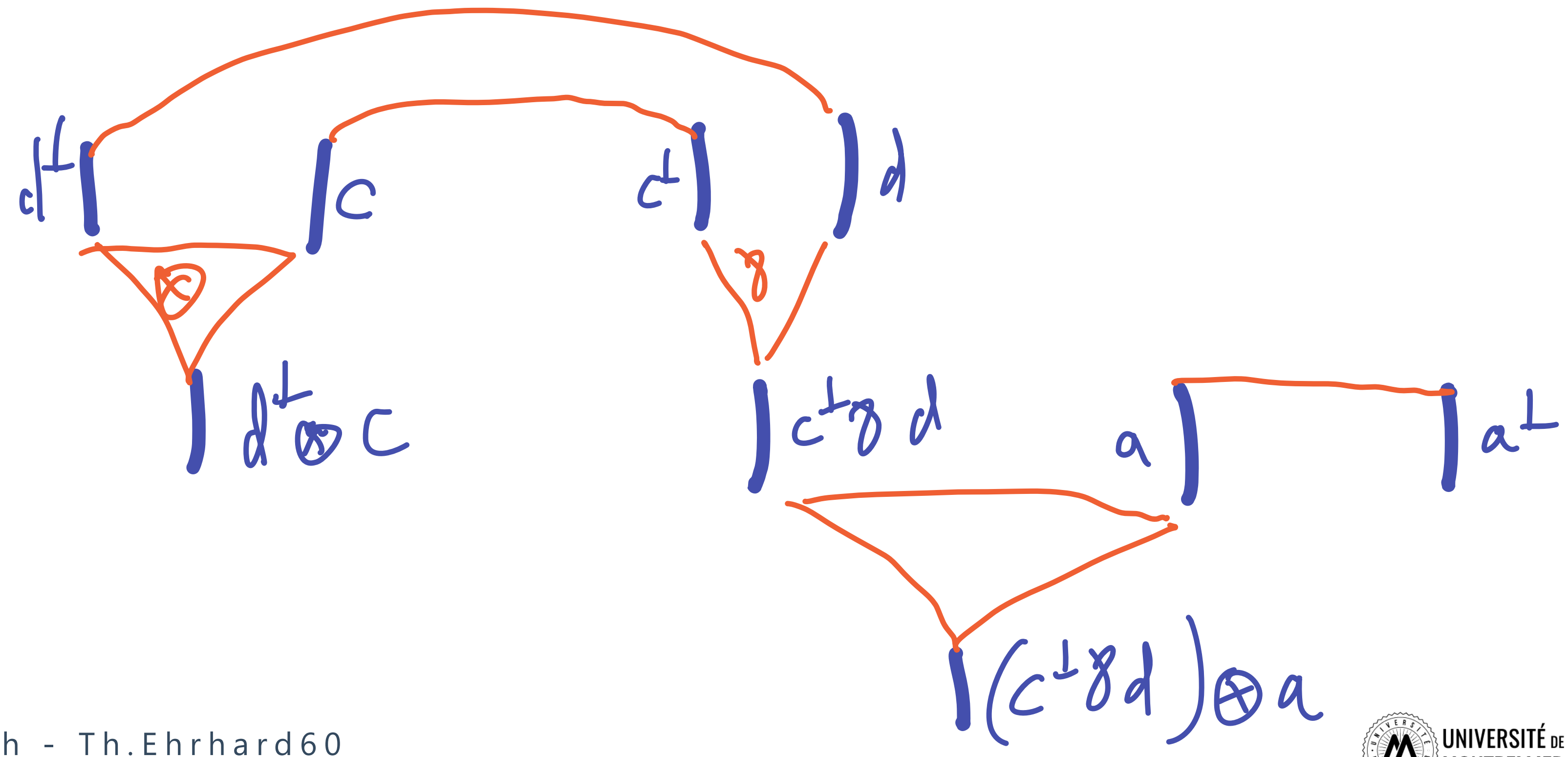
Proof structures (RnB proof nets, my favourites)

A natural pile-up of links:



Proof nets : criterion à la Danos-Reigner

General

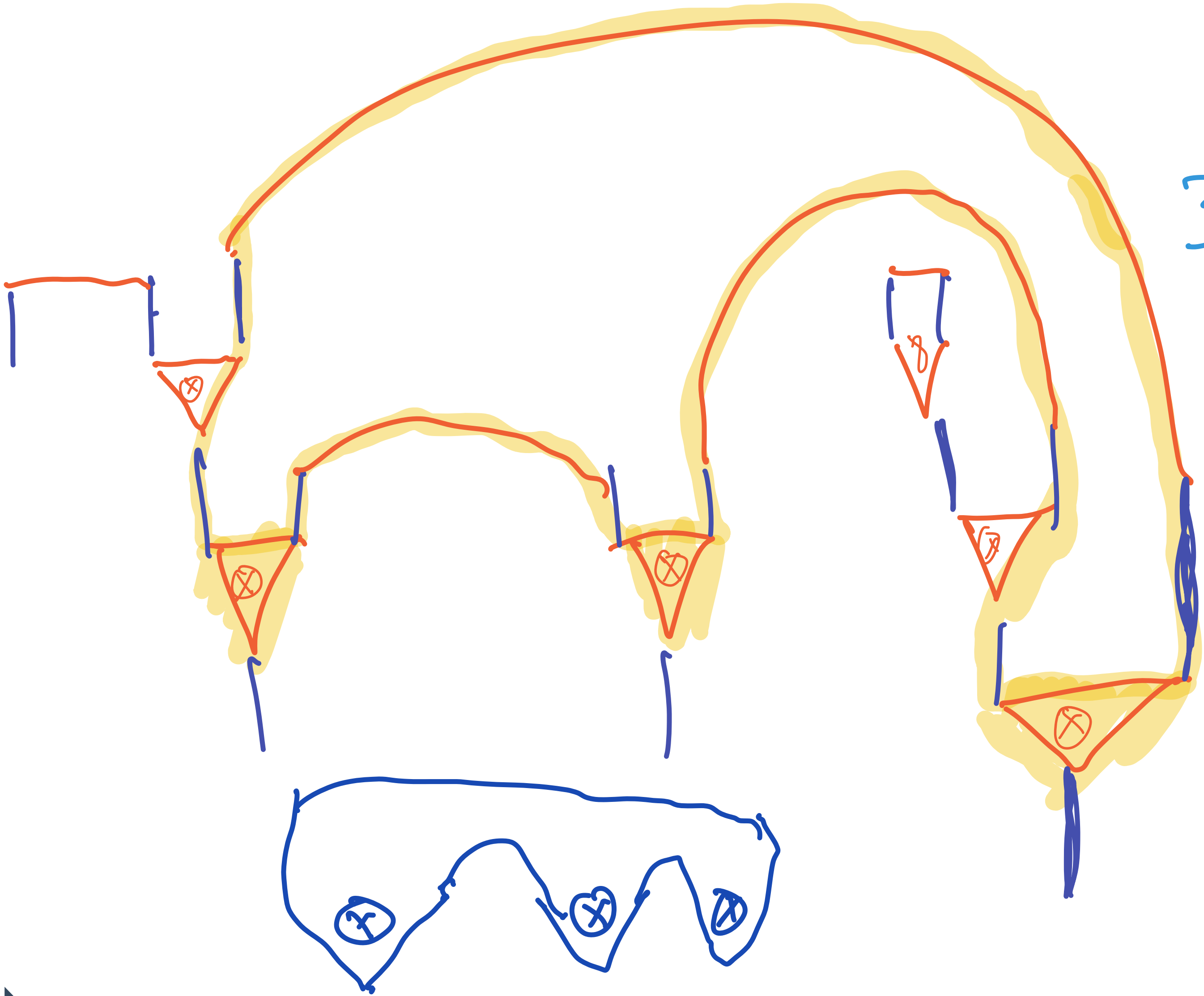
Every alternate elementar cycle contains a chord

Here (because of the shape of links)

No alternate elementary cycle

(∃ an alternate elementary path
between any two premises
in order to exclude the Mix rule)

Proof structures and nets

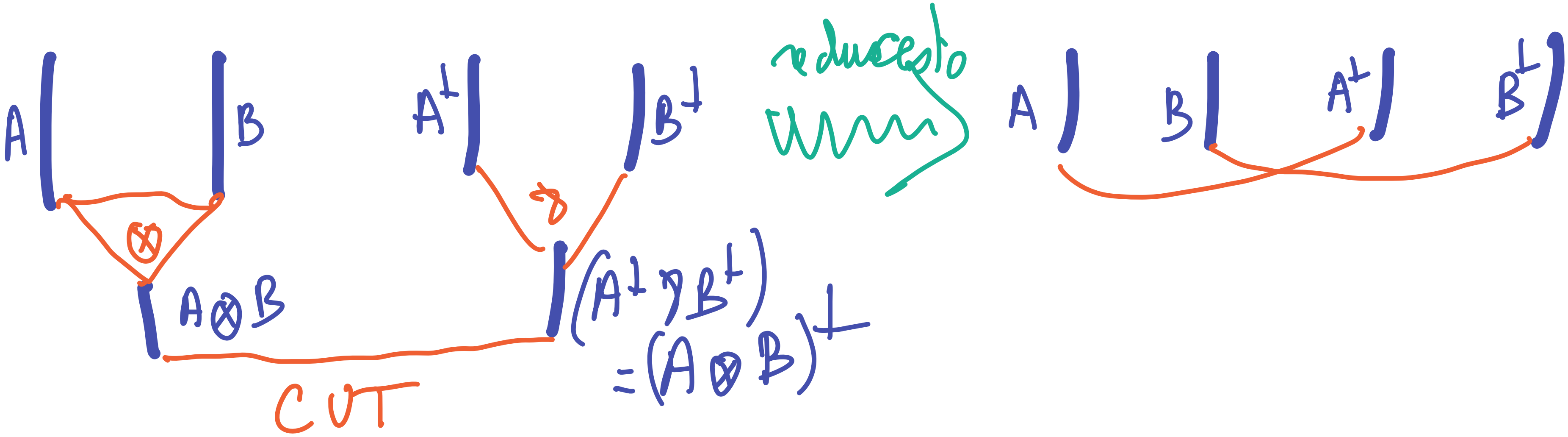


If one of the 3 $\otimes$ in the cycle becomes $\parr$, proof NET

Cut-elimination

In MLL(+mix) that's clearly a terminating process

preserves the absence of ae-cycle



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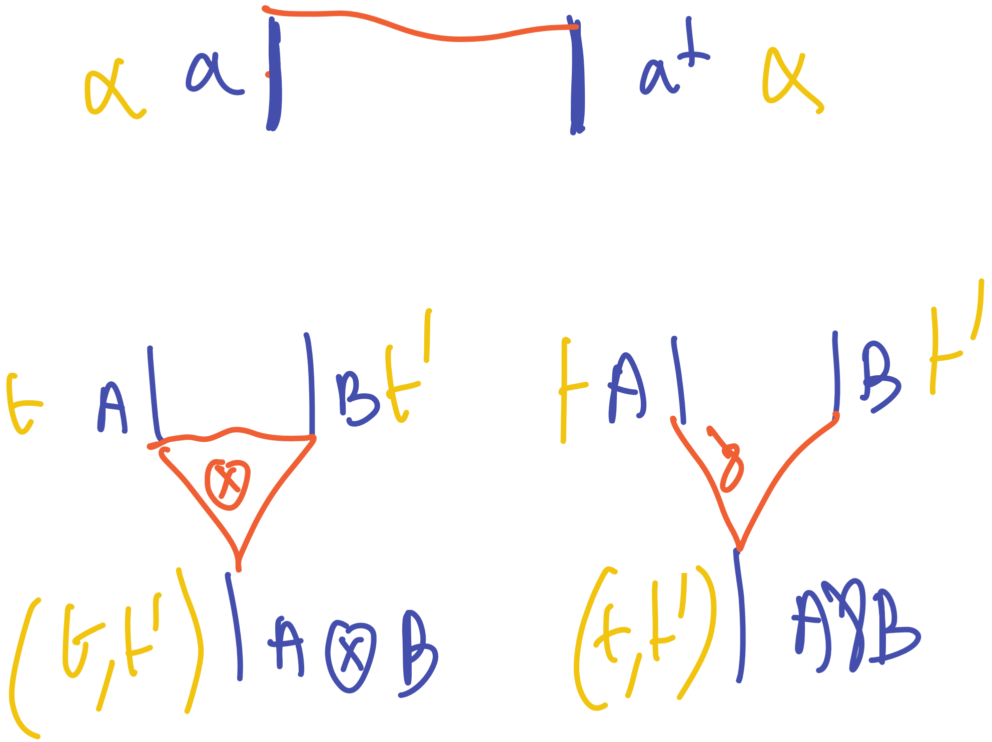
Interpreting a proof (net) of X as clique of the corresponding coherence space

Experiments (Girard 1987 LL paper) slightly revisited

EXPERIMENTS À LA GIRARD BUT UP-SIDE DOWN



a coherence space

label token

$A \quad \alpha \in |A|$

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CUTS?

$$\langle x_1 \quad x_n \mid \underbrace{K \qquad K^{\perp}}\mid y_1 \quad y_n \rangle$$

ok for $x_1 = y_1 \quad y_2 = y_2 \quad \ldots \quad x_n = y_n$

~~[illegible]~~ RESULT of an experiment

The tuples collected on conclusions for the succeeding experiments

All the results of all the succeeding experiments in π are $[\![\pi]\!]$

1) a clique of $C_1, C_2, \ldots, C_p$

2) preserved by cut elimination

« Soundness »
(Girard 1987)

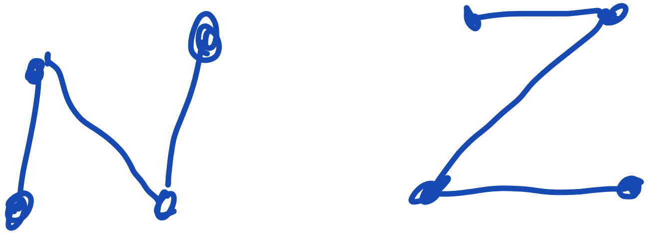
suggestion

incorrect (cycle) => there are two experiments who results are incoherent

Remarks

If two experiments differ ~~on~~ some where then they are coherent on some conclusion.

IDEA: extending a path up incoherent $\smile$ down $\frown$ coherent

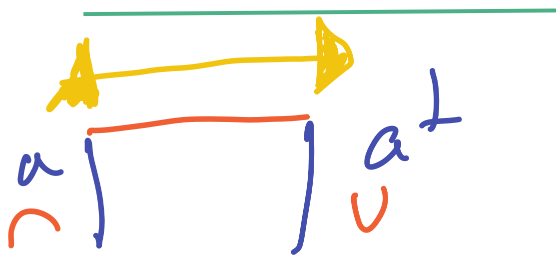


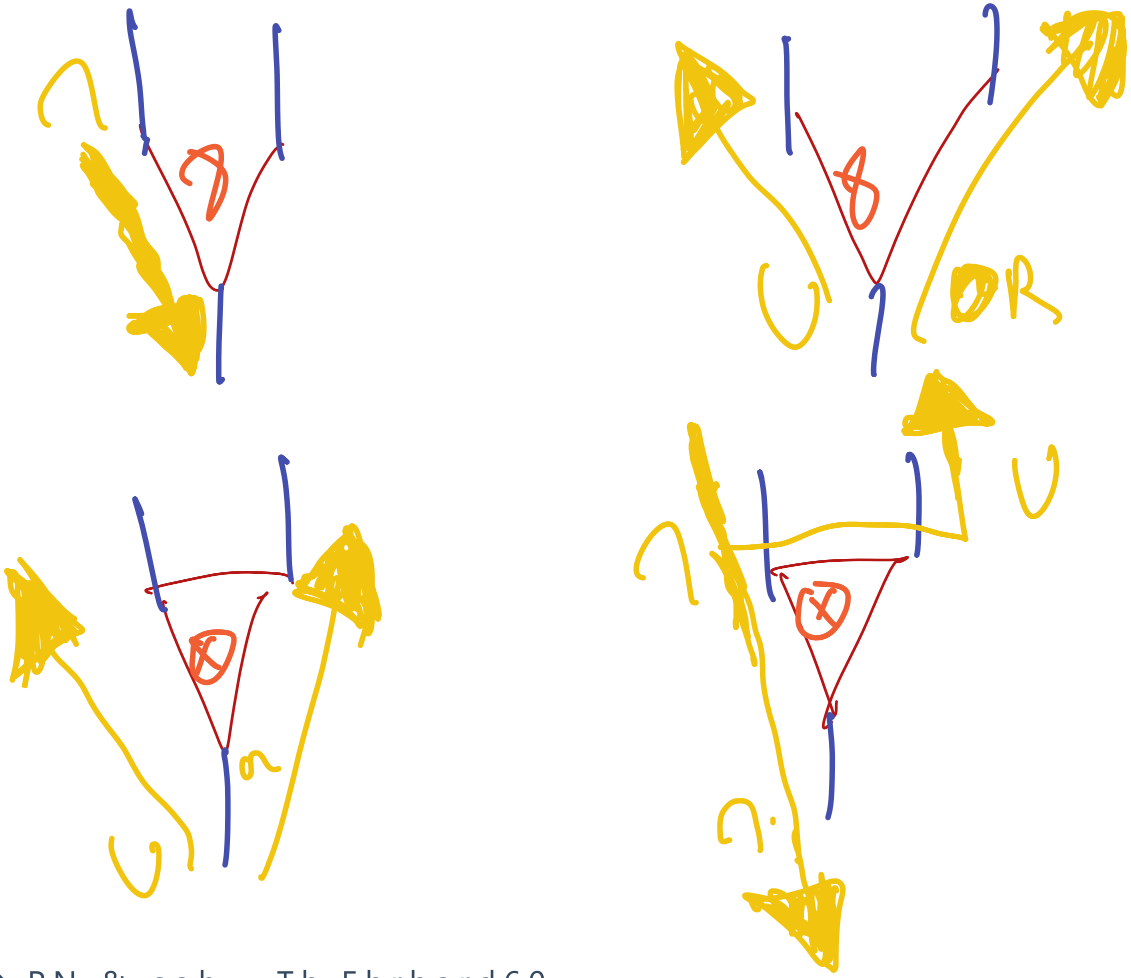
$$N^{\perp} = Z$$

Remarks

≠ incohérent ↑ up

≠ cohérent ↓ down





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« Completeness »

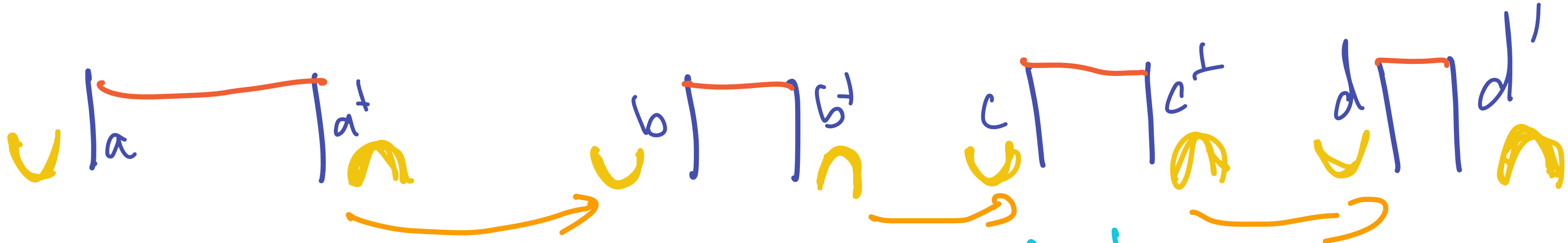
(Inria report Retoré 1994 paper in MSCS 1997)
presented in 1995 for Dana Scott honorary degree in Darmstadt

A remark

We need a coherence space N such that there are
- two coherent tokens in N
- Two incoherent tokens in N (coherent in $N^{\perp}$)

- We take N to be isomorphic to his negation

Given an ordered sequence of axioms
(with a first and a second conclusion)
There exist two diferent experiments such that



obvious : take a token that is $\frown$ [illegible] $\smile$ with the one you already have

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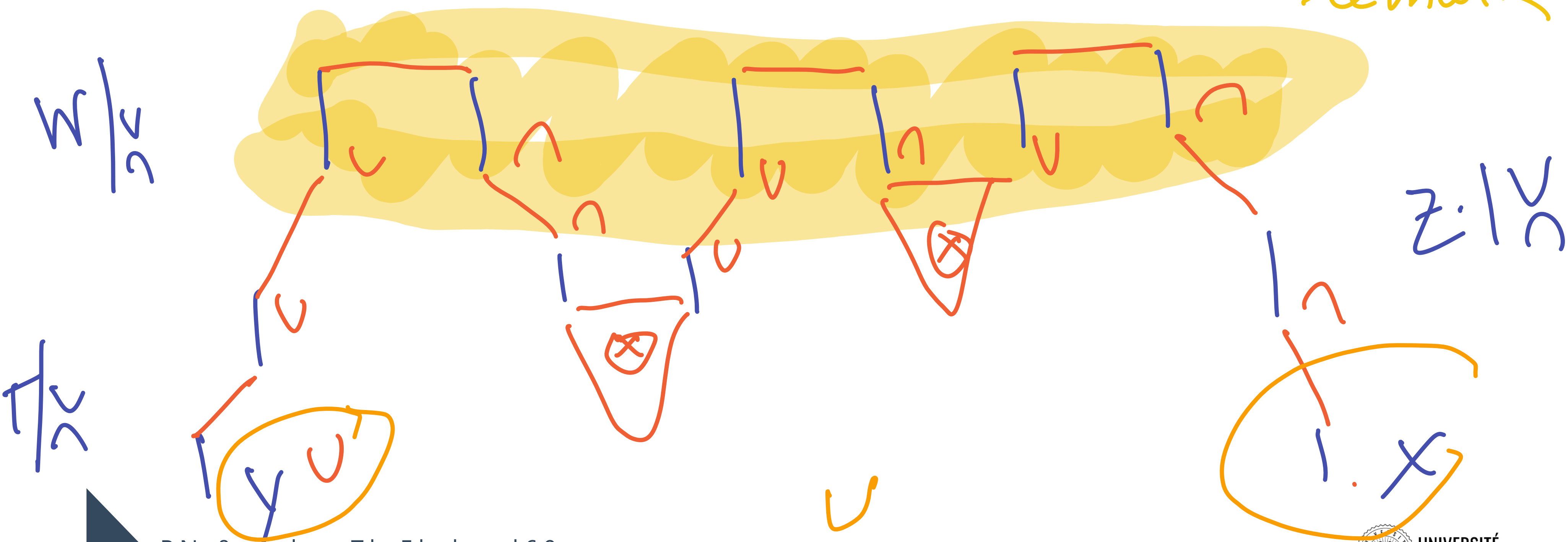
Lemma

Induction on the proofnet using sequentialisation

In a proof NET when there exist an ae path from a conclusion X to a conclusion Y one may found two experiments such that they are coherent on X, incoherent on Y. and $\smile\frown$ on any other conclusion

as in the remark



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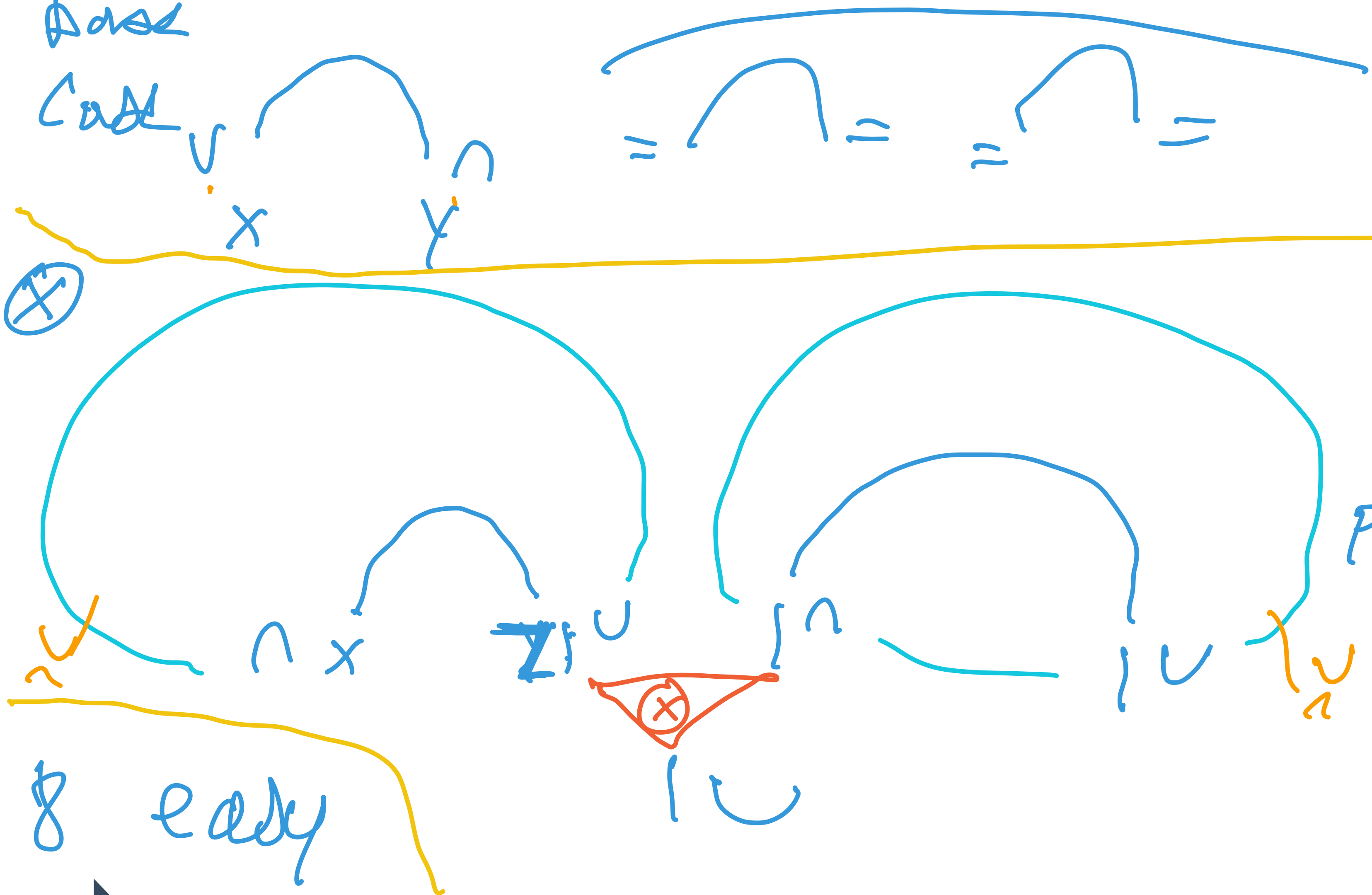
Completeness

When a proof structure is not correct it is possible to find two expériment such that the results are incoherent on all conclusion na strictly incoherent on one conclusion — hence incoherent w.r.t. the par of all the conclusions.

Completeness / proof

base case

others



part on one side ok

⅋ easy

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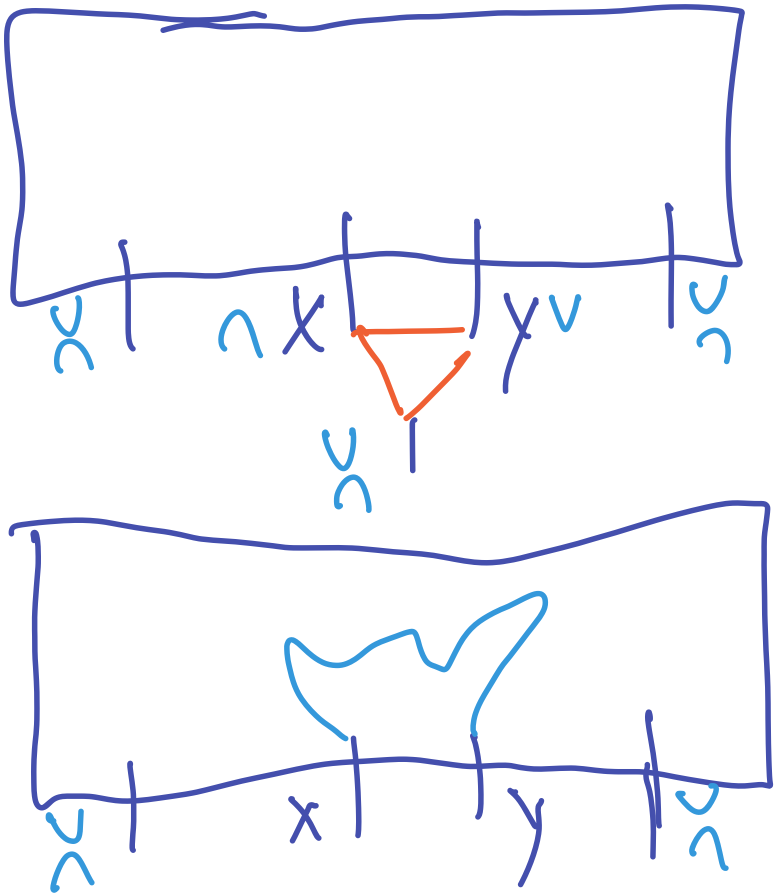
Completeness / proof



incorrect ? yes ok

no

Remove x? y?

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Deadlock freeness of the reduct

(Retoré 1994)

Deadlock free-ness

(Asperti terminology i think)

No loop in the reduced proof STRUCTURE.

An incorrect

~~A~~ proof structure may well reduce to a proof net.

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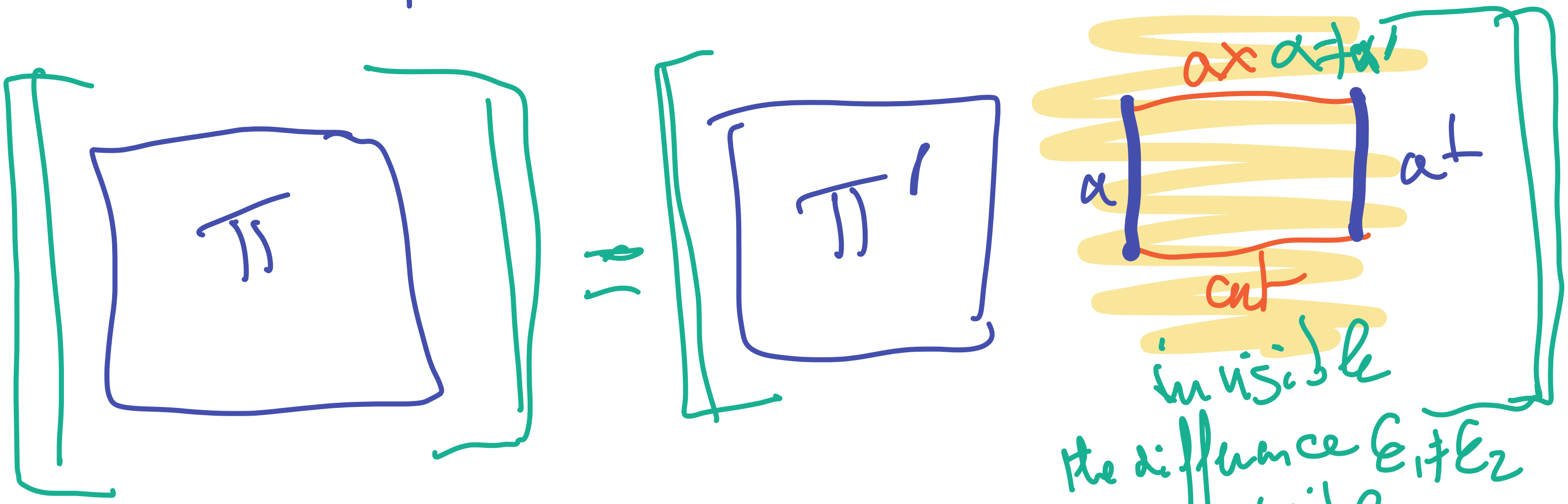
Characterisation

Two ~~experiment~~ different experiments yield the same result:
there will be a loop in the normal form.

The interpretation is preserved by cut-elimination



the difference $\varepsilon_1 \neq \varepsilon_2$
is invisible

Extensions to MELL and MALL

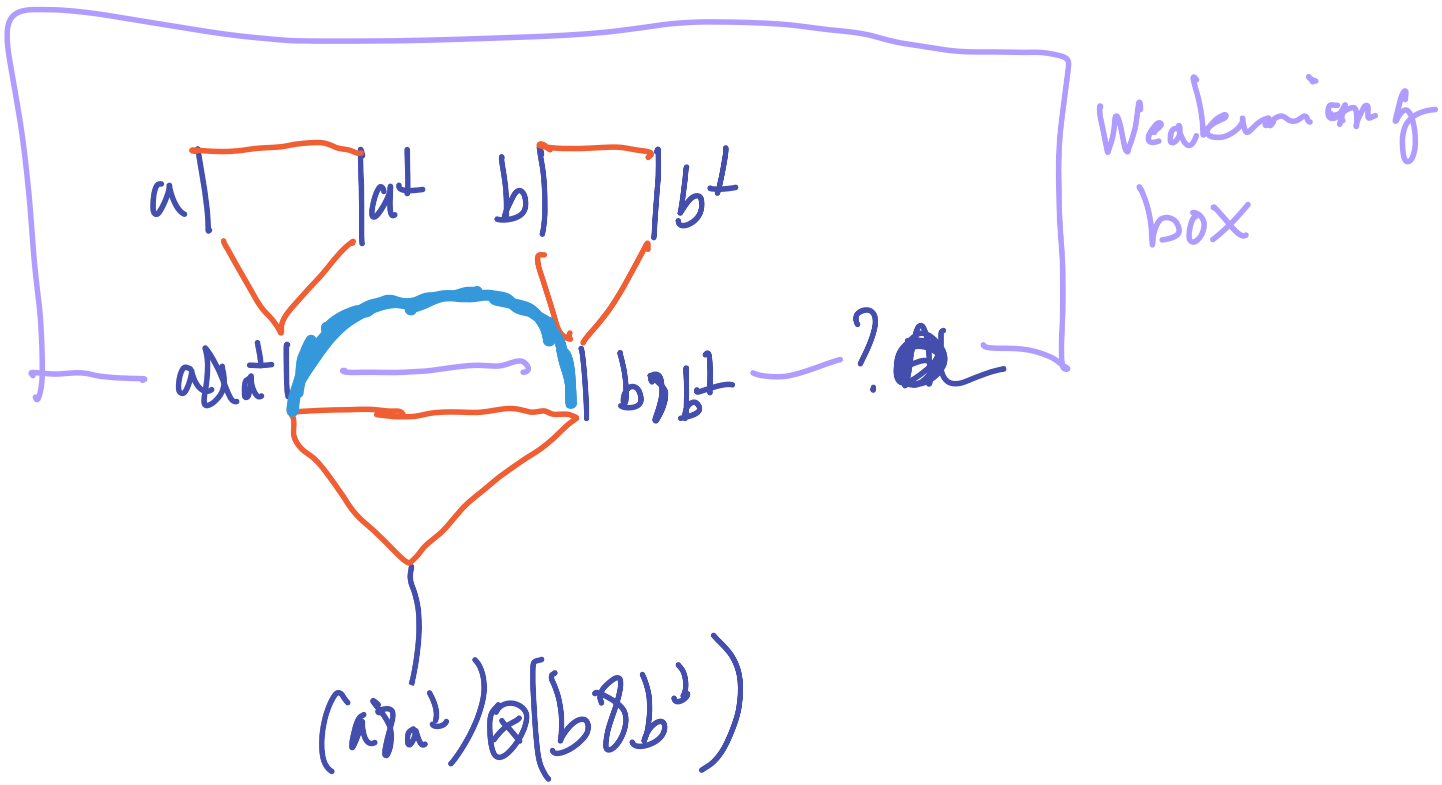
Pagani Tasson

Not easy to extend, because of mix

While the criterion for boxes is:
1) Check whether the inside of a boc is correct, and
2) Replace the box with a kind of n-ary axioms yielding the conclusions of the box, is the out side correct?

Example by Michele Pagani

Michele will explain that tomorrow. (much better than I can)

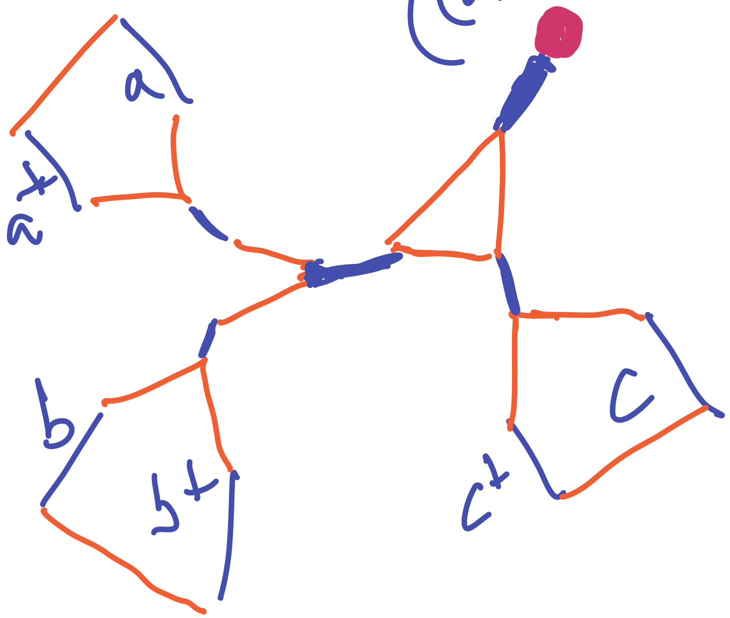


Conclusion ⊗ perspectives

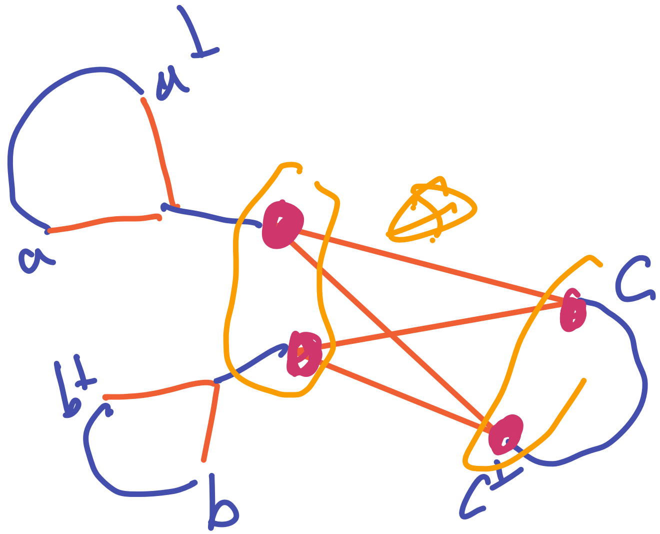
From RnB graphs to handsome proof nets

FOLD / UNFOLD

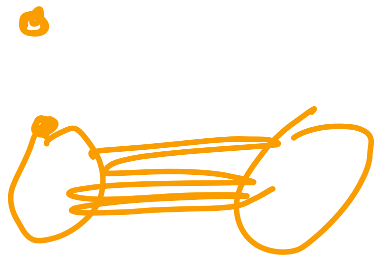
$((a \parr a^\perp) \parr (b \parr b^\perp)) \otimes (c \parr c^\perp)$



$\rightsquigarrow$

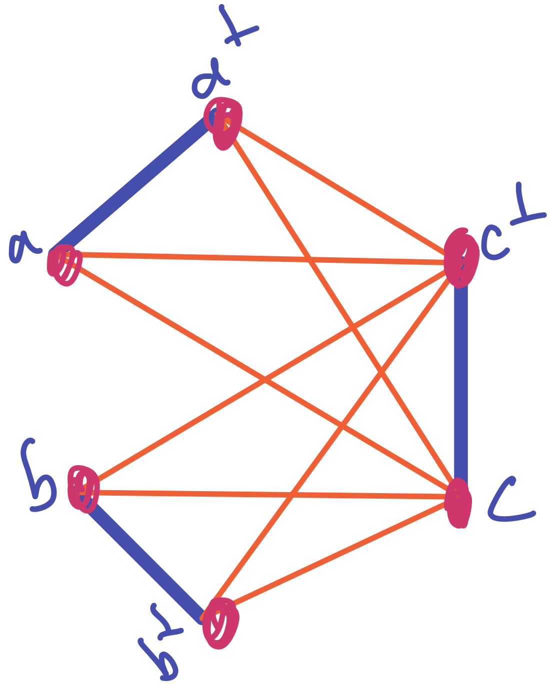


cographs



From RnB graphs to handsome proof nets

COGRAPH (FORMULA) + PERFECT MATCHING (AXIOMS)



$$(c \parr c^\perp) \otimes \left((a \parr a^\perp) \parr (b \parr b^\perp)\right)$$

Criterion for handsome proof nets Retoré & Ehrhard

AXIOMS : PERFECT MATCHING

CONCLUSION: A COGRAPH DESCRIBING THE FORMULA

THERE IS A CHORD ON EVERY ALTERNATE ELEMENTARY CYCLE

From Thomas' web page:

Thomas Ehrhard. *A new correctness criterion for MLL proof nets.* 2014. Accepted at LICS'14. This criterion was first published by C. Rétoré in TCS 294(3):473-488, 2003. I rediscovered it independently (my presentation is slightly more general) and I am convinced that it is worth being further studied. pdf.

unusual! THANKS!

Question (discussed with Thomas)

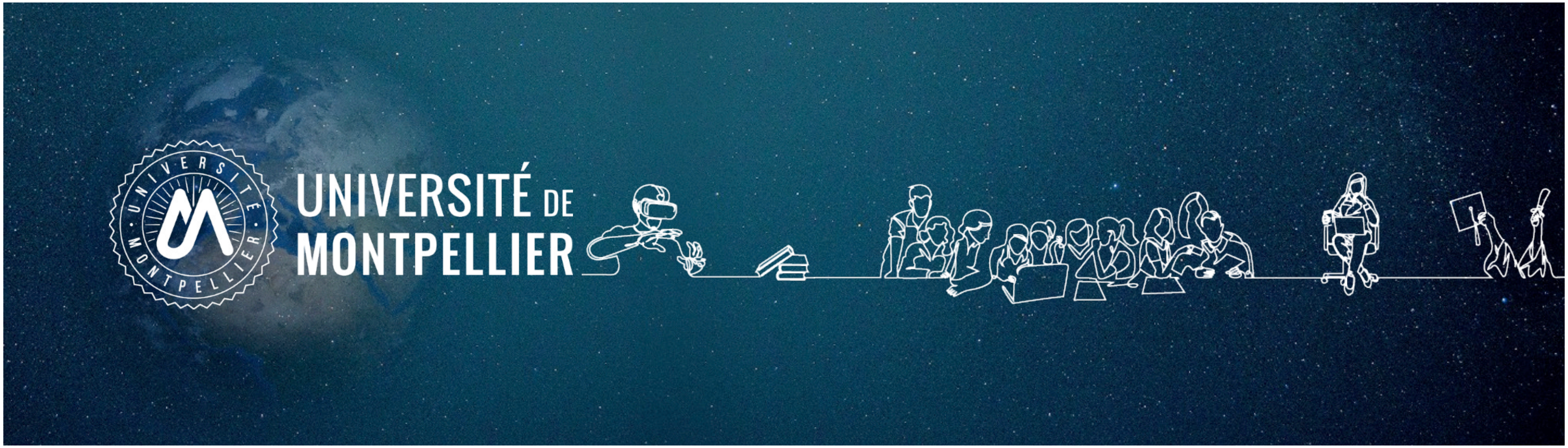
Intuition

- The semantic criterion with coherence spaces
- The handsome criterion

should be related / we know they are equivalent ...

So far no argument, just an intuition.

could we find out why?



Happy birthday Thomas