# Proof nets without links

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# Plan

# 1. Multiplicative linear logic

$$\mathcal{G} ::= P \quad | \quad \mathcal{G} \wp \mathcal{G} \quad | \quad \mathcal{G} \otimes \mathcal{G} \quad | \quad \mathcal{G} \multimap \mathcal{G}$$

De Morgan laws:

$$\begin{array}{rcl} (A^\perp)^\perp & \equiv & A \\ (A \wp B)^\perp & \equiv & (B^\perp \otimes A^\perp) \\ (A \otimes B)^\perp & \equiv & (B^\perp \wp A^\perp) \end{array}$$

Deductive system:

$$\frac{\Theta, A, B, \Gamma \vdash \Delta}{\Theta, B, A, \Gamma \vdash \Delta}\ XT_l \qquad \text{exchange} \qquad \frac{\Gamma \vdash \Delta, A, B, \Psi}{\Gamma \vdash \Delta, B, A, \Psi}\ ET_r$$

$$\frac{axiom}{A \vdash A} \qquad\qquad \frac{\Gamma \vdash \Delta, X \quad X, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

$$\frac{\Gamma \vdash \Delta, A}{A^{\perp}, \Gamma \vdash \Delta}\ \perp_l \qquad \text{negation} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\ \perp_r$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\ \otimes_l \qquad \text{conjunction} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B}\ \otimes_r$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \wp B}\ \wp_r \qquad \text{disjunction} \qquad \frac{A, \Gamma \vdash \Delta \quad B, \Gamma' \vdash \Delta'}{A \wp B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \wp_l$$

$$\frac{B, \Gamma \vdash \Delta \quad \Gamma'\Delta', A}{A \multimap B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \multimap_l \qquad \text{implication} \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \multimap B}\ \multimap_r$$

# 2. An equivalent but simpler calculus

For the following reasons:

- de Morgan laws
- implication is definable:

$$A \multimap B \equiv A^{\perp} \wp B$$

- complex axioms are derivable

The language can be restricted to:

$$\mathcal{F} ::= P \quad | \quad P^{\perp} \quad | \quad \mathcal{F} \wp \mathcal{F} \quad | \quad \mathcal{F} \otimes \mathcal{F}$$

with the following rules:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, B, A}\, XT \qquad \frac{\vdash \Gamma, A}{\vdash A, \Gamma}\, XC$$

$$\frac{\mathit{axiom}}{\vdash p, p^{\perp}} \qquad \frac{\vdash \Gamma, X \quad \vdash X^{\perp}, \Gamma'}{\vdash \Gamma, \Gamma'}\, cut$$

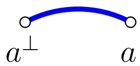
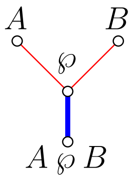
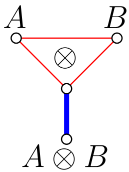
$$\frac{\vdash \Gamma, A \quad \vdash B, \Gamma'}{\vdash \Gamma, A \otimes B, \Gamma'}\, \otimes \qquad \frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \wp B, \Gamma'}\, \wp$$

# 3. Proof nets with links

- abstract mathematical structure independent from the names of the formulae
- two equivalent definitions:
  - derivational (generative, inductive, existential)
  - representational (model theoretic, global, universal)
- contexts are not copied
- quotient proof trees by permuting rules
- more efficient cut-elimination procedure (local, sharing)
- allows new proof search (parsing?) techniques

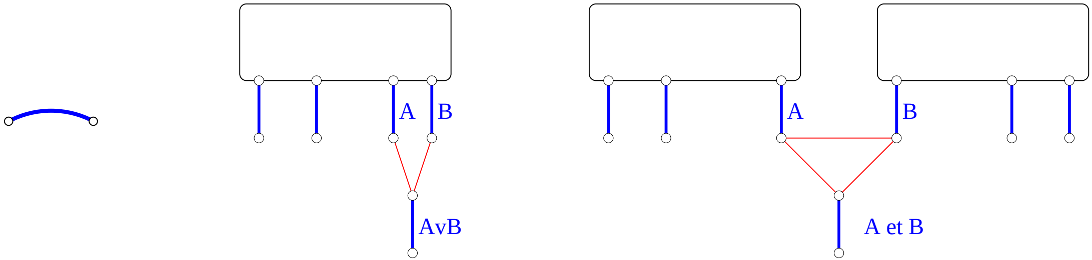
- Formulae : blue edges (perfect matching)
- Connectives : red edges
- Criterion : no *Æ* cycle
  ***Æ* path/cycle alternate elementary path/cycle:**
  **a blue edge, a red edge, a blue edge, a red edge, .... not crossing itself**

## Liens

| Name | axiom link | par link | tensor link |
|---|---|---|---|
| Premisses | none | $A$ et $B$ | $A$ et $B$ |
| edge bicolored graph |   |   |   |
| Conclusions | $a$ et $a^\perp$ | $A \wp B$ | $A \otimes B$ |

# 3.1. From proofs to graphs

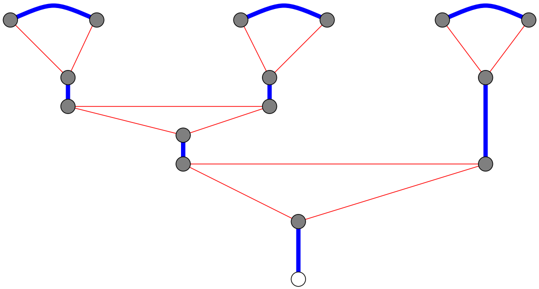
Proof $\longrightarrow$ graphs endowed with a perfect matching (by induction)
Grosso modo: sub-formula tree + axioms.




Notice that

- no Æ cycle
- two vertices are always connected via an Æ path
  one does not have MIX : $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ MIX (A \otimes B) \multimap (A \wp B)$

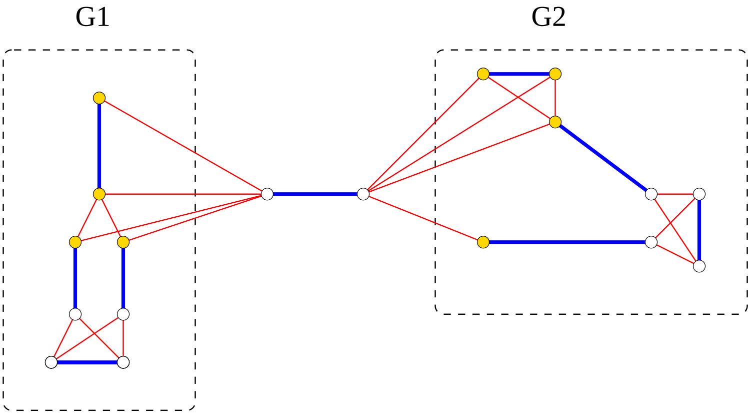
Example $((a \wp a^\perp) \otimes (b \wp b^\perp)) \otimes (c \wp c^\perp)$:



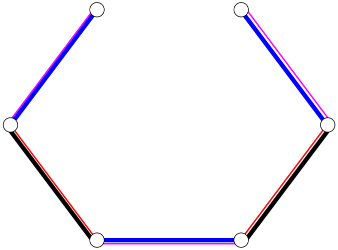
## 3.2. From graphs to proofs

**3.2.1. Lemma** The following properties are equivalent:

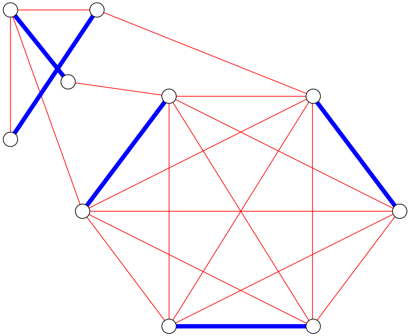
1. no Æ cycle
2. the perfect matching is unique
3. the graph recursively contains a blue bridge

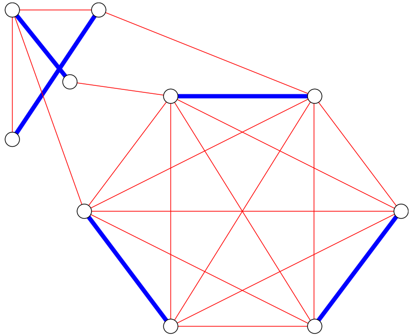


- $\neg 2 \Rightarrow \neg 1$ easy



- $\neg 1 \Rightarrow \neg 2$ easy

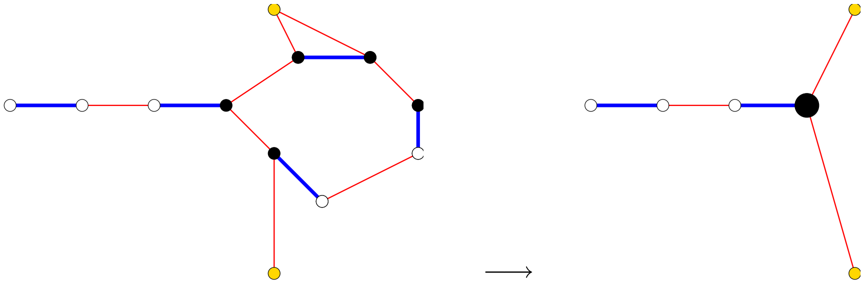




- $3 \Rightarrow 1$ obvious
- $1 \Rightarrow 3$

  extend as much as possible an Æ path

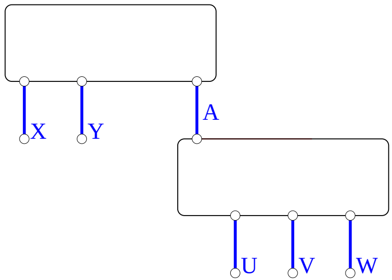
  - pending bridge
  - loop



- Æ path in the reduced graph $\longrightarrow$ Æ path in the initial graph
- bridge in the reduced graph $\longrightarrow$ bridge in the initial graph
- by induction on the size of the graph

### 3.2.2. Proof net → proof

- One can assume w.l.o.g. that all conclusions of the proofs are conjunctions: (final disjunctions are easy to handle).
- One takes off the final conjunctions, except the red edges between the two premisses
  there exists a blue bridge (lemma)
  - If the bridge is one of the premisses of a conclusion conjunction the conjunction rule applies




  - Otherwise the blue bridge is like this:
    By induction hypothesis we have:

- a proof $\delta$ of $\vdash A^{\perp}, U, V, W$ one axiom of which is $\vdash A, A^{\perp}$

- a proof $\gamma$ of $\vdash X, Y, A$.

$$\dfrac{\dfrac{\overset{\gamma}{\vdash X, Y, A}}{\vdash [A^{\perp} := X, Y], A} \qquad \vdash \cdots \qquad \vdash \cdots}{\overset{\delta}{\vdash [A^{\perp} := X, Y], U, V, W}}$$

# 4. Proof nets without links

Idea: identifying more proofs, but not all of them.
Algebraic properties of the connectives:

- associativity
  - $A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C$
  - $A \wp (B \wp C) \equiv (A \wp B) \wp C$
- commutativity
  - $A \otimes B \equiv B \otimes A$
  - $A \wp B \equiv B \wp A$

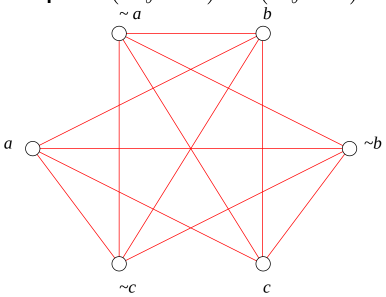
Sequent or disjunction of the formulae in a sequent $\rightarrow$ cograph

- $p \longrightarrow$ vertex (no edges)
  vertices = propositional variables

- $A \wp B \longrightarrow$ disjoint union

- $A \otimes B \longrightarrow$ series composition

Example: $(a \wp a^{\perp}) \otimes (b \wp b^{\perp}) \otimes (c \wp c^{\perp})$



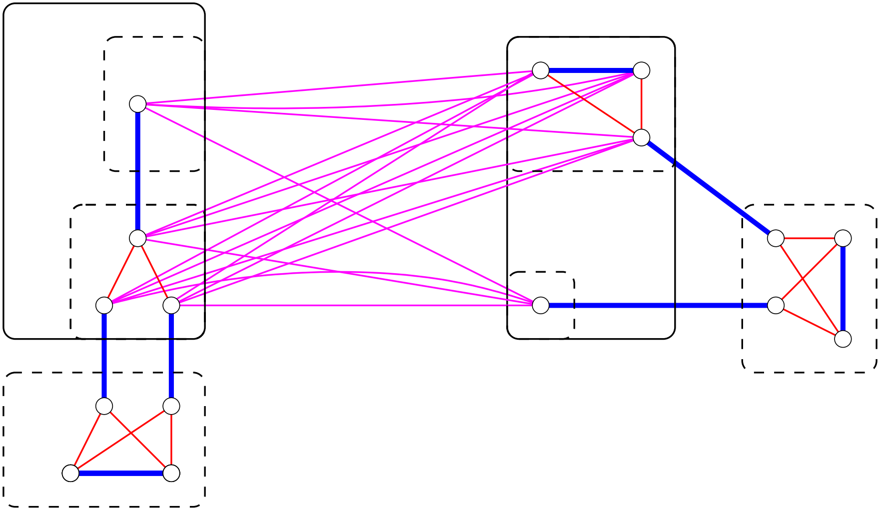
## 4.1. From proofs to graphs

Inductive definition:

- axiom $\longrightarrow$ blue edge
- $\wp$ rule $\longrightarrow$ no effect

- $\otimes$ rule $\longrightarrow$ we select a family of connected components in each cograph and we compose them in the series mode

A=p v (a et (b v c)) B=(x et (y v z)) v t



Proof net = cograph + perfect matching

- every Æ cycle contains a chord
- between any two vertices there exists a chordless Æ path (no MIX)

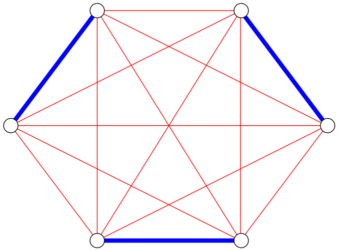
Easy to prove by induction on the number of rules.

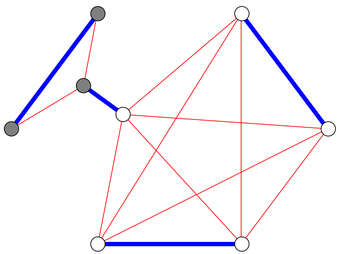
## 4.2. From graphs to proofs

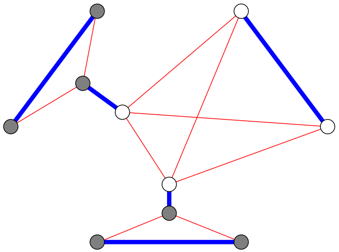
One way to prove this is to go back to the previous case, little by little, by considering intermediate structures

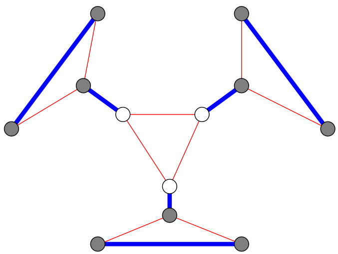
- axioms
- subformulae trees of the conlusions
- cographs whose vertices are conclusions

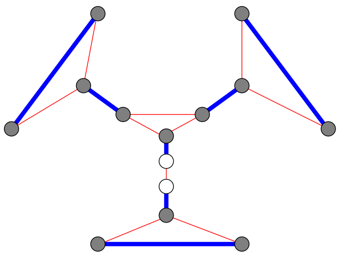
$(a \wp a^{\perp}) \otimes (b \wp b^{\perp}) \otimes (c \wp c^{\perp})$ (correct)

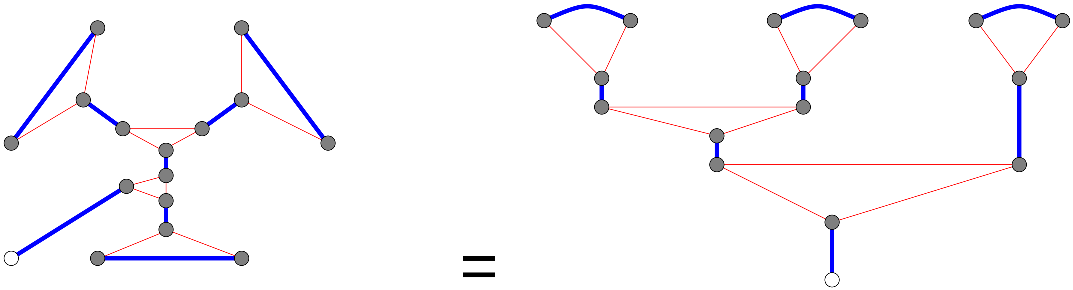












- The transformation and its inverse preserve the criterion:

  - every Æ cycle contains a chord
  - between any two vertices, there exists a chordless Æ path

- Since for a proof net with links there can not be any chord in an Æ path, we have:

- every graph satisfying the criterion corresponds to (at least) one proof.

## 4.3. A definition of proofs by proof net rewriting

Every proof nets is obtained from the complete bicolored graph

$$\wedge_{i \in I}(a_i \wp a_i^{\perp}) \quad I \text{ multiset}$$

by the following rewrite rule (modulo commutativity and associativity):

$$X \otimes (Y \wp Z) \rightarrow (X \otimes Y) \wp Z$$

If MIX is allowed, add the following rule:

$$X \otimes Y \rightarrow X \wp Y$$

# 5. Proof Nets for Lambek Calculus

## 5.1. Reminder: Proof Nets with Links (Abrusci & Maringelli, 1998, JoLLI)

Red egdes of links are directed.
Criterion:

- the underlying undirected graph is a correct MLL proof net
(every Æ path contains a chord)
- there always exist an Æ path
from the right premise of a disjunction to its left premise
- there is a black cyclic order between the conclusions,
and for every black arc $x \to y$ there is a directed Æ path from $y \longrightarrow x$

Idea: similar black arcs for $\wp$ links, that we unfold.

## 5.2. Non commutative proof nets without links

### 5.2.1. Directed cographs

- Disjoint union

- Directed series composition between $(V, A)$ and $(V', A')$

  - vertices $V \uplus V'$

  - arcs= $A \cup A' \cup (V \times V')$ (and not $A \cup A' \cup (V \times V') \cup (V' \times V)$)

  - $V$ is called the first **component** and $V'$ the second component of the directed series composition.

**5.2.2. Cyclic orders** A total cyclic order is a ternary relation $C(x, y, z)$ (moving from $x$ to $z$ in the right direction, one meets $y$):

- $C(x, y, z) \Rightarrow C(y, z, x)$
- $C(x, y, z) \wedge C(y, u, z) \Rightarrow C(x, y, u) \wedge C(x, u, z)$
- $C(x, y, z) \vee C(z, y, x)$

The **interval** $[a, b]$ is $\{z | C(a, z, b)\}$.

### 5.2.3. K-graph of a formula

$K(a) \quad a \in A:$

**vertices:** $\{a\}$

$N$**-arcs** $\emptyset$

$R$**-arcs** $\emptyset$

$K(X \otimes Y)$

**vertices:** $V(F) = V(X) \uplus V(Y)$

$N$**-arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y))$

$R$**-arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y)) \uplus V(X) \times V(Y)$

$K(X \wp Y)$

**vertices:** $V(F) = V(X) \uplus V(Y)$

$R$**-arcs** $R(K(F)) = R(K(X)) \uplus R(K(Y))$

$N$**-arcs** $N(K(F)) = N(K(X)) \uplus N(K(Y)) \uplus V(X) \times V(Y)$

Total order on the atoms.
Underlying graph: Komplet graph.

### 5.2.4. Sequent structure $\vdash A_1, ..., A_n$

Cyclic order on the family of $n \geq 1$ $K$ graphs $K(A_1), ..., K(A_n)$, materialised by a black arc from the last vertex in $K_i$ to the first vertex in $K_{i+1}[n]$ **whose main connective is an R series-composition.** ($\otimes$).

In particular, if there is a single $K_1$ then there is an edge from the last vertex in $K_1$ to the first vertex in $K_1$.

Notice that one has a unique Hamiltonian circuit (total order + cyclic order).

The graph is invariant with respect to associativity, but makes a difference between comma and $\wp$. (Otherwise the criterion is slightly more complicated).

## 5.3. Cyclic proof nets without links

Proof structure:

- a sequent structure (set of cyclically ordered K graphs)
- a set of B-edges which are a perfect matching.

Correctness criterion:

**MLL proof-net** the underlying undirected $BR$-graph is an MLL proof-net, that is to say:

**acyclicity** Every alternate elementary cycle contains a chord.

**connectedness** There exists a chordless alternate elementary path between any two vertices.

**Hamiltonian adequacy** $B$-edges are adequate to the Hamiltonian circuit: that is whenever $aBa'$, $bBb'$and $H(a, b, a')$ one has $H(a, b', a')$ as well.

### 5.3.1. Alternative definition

One black arc per $\wp$ composition (from the last atom of the first component to the first atom of the second component) $L(A)$ Leight.

The Hamiltonian circuit makes use of all black arcs.

Advantage: less arcs, and there is no difference between commas and $\wp$.

Disadvantage: in this case, the criterion has a supplementary condition: each R series composition is an interval of the cyclic order.

# 6. Conclusion

- Pomset logic Similar notion of proof net for an extension with a non commutative and autodual connective (further studied by Alessio Guglielmi as a term calculus). This connective corresponds to directed series composition.

- Redundancy? Can the criterion and/or structure can be simplified?

- What about cuts? Undirected even in the non commutative case.

- Mixed system What about partially commutative linear logic (de Groote, 96) and its extension, Non-Commutative logic (Ruet 1999, Abrusci-Ruet 2000)

- Algorithms Can proof nets without links be used for optimised proof net construction as in Moot's proof search?