Proof nets:
cographs and perfect matchings

Christian Retoré

LaBRI (C.N.R.S. & Université de Bordeaux)

INRIA Bordeaux Sud-Ouest

Plan

1. Proof Nets for Multiplicative Linear Logic

$$\mathcal{G} ::= P \quad | \quad \mathcal{G} \vee \mathcal{G} \quad | \quad \mathcal{G} \wedge \mathcal{G} \quad | \quad \mathcal{G} \multimap \mathcal{G}$$

De Morgan laws:

$$\begin{array}{rcl} (A^{\perp})^{\perp} & \equiv & A \\ (A \vee B)^{\perp} & \equiv & (B^{\perp} \wedge A^{\perp}) \\ (A \wedge B)^{\perp} & \equiv & (B^{\perp} \vee A^{\perp}) \end{array}$$

Deductive system:

$$\frac{\Theta, A, B, \Gamma \vdash \Delta}{\Theta, B, A, \Gamma \vdash \Delta} XT_l \qquad \text{exchange} \qquad \frac{\Gamma \vdash \Delta, A, B, \Psi}{\Gamma \vdash \Delta, B, A, \Psi} ET_r$$

$$\frac{axiom}{A \vdash A} \qquad \frac{\Gamma \vdash \Delta, X \quad X, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} cut$$

$$\frac{\Gamma \vdash \Delta, A}{A^\perp, \Gamma \vdash \Delta} \perp_l \qquad \text{negation} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A^\perp} \perp_r$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge_l \qquad \text{conjunction} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B} \wedge_r$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \vee_r \qquad \text{disjunction} \qquad \frac{A, \Gamma \vdash \Delta \quad B, \Gamma' \vdash \Delta'}{A \vee B, \Gamma, \Gamma' \vdash \Delta, \Delta'} \vee_l$$

$$\frac{B, \Gamma \vdash \Delta \quad \Gamma' \Delta', A}{A \multimap B, \Gamma, \Gamma' \vdash \Delta, \Delta'} \multimap_l \qquad \text{implication} \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \multimap B} \multimap_r$$

1.1. An equivalent but simpler calculus

For the following reasons:

- de Morgan laws

- implication is definable:

$$A \multimap B \equiv A^{\perp} \vee B$$

- complex a xioms are derivable

The language can be restricted to:

$$\mathcal{F} ::= P \quad | \quad P^{\perp} \quad | \quad \mathcal{F} \vee \mathcal{F} \quad | \quad \mathcal{F} \wedge \mathcal{F}$$

with the following rules:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, B, A}\, XT \qquad \frac{\vdash \Gamma, A}{\vdash A, \Gamma}\, XC$$

$$\frac{\quad}{\vdash p, p^{\perp}}\, axiom \qquad \frac{\vdash \Gamma, X \quad \vdash X^{\perp}, \Gamma'}{\vdash \Gamma, \Gamma'}\, cut$$

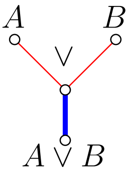
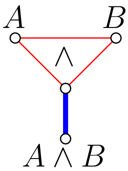
$$\frac{\vdash \Gamma, A \quad \vdash B, \Gamma'}{\vdash \Gamma, A \wedge B, \Gamma'}\, \wedge \qquad \frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \vee B, \Gamma'}\, \vee$$

1.2. Proof nets with links

- abstract mathematical structure independent from the names of the formulae
- two equivalent definitions:
  - derivational (generative, inductive, existential)
  - representational (model theoretic, global, universal)
- contexts are not copied
- quotient proof trees by permuting rules
- more efficient cut-elimination procedure (local, sharing)
- allows new proof search (parsing?) techniques

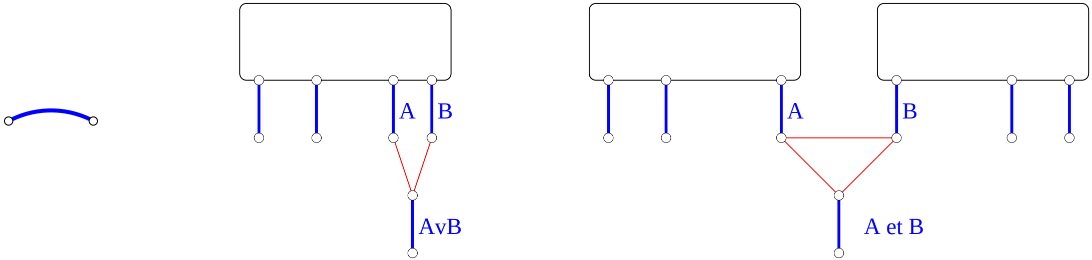
- Formulae : blue edges (perfect matching)
- Connectives : red edges
- Criterion : no Æ cycle
  Æ path/cycle alternate elementary path/cycle:
  a blue edge, a red edge, a blue edge, a red edge, ... not crossing itself

Liens

Name	axiom link	par link	tensor link
Premisses	none	A et B	A et B
edge bicolored graph			
Conclusions	a et $a^\perp$	$A \vee B$	$A \wedge B$

1.2.1. From proofs to graphs with a perfect matching

Grosso modo: subformula tree + axioms.

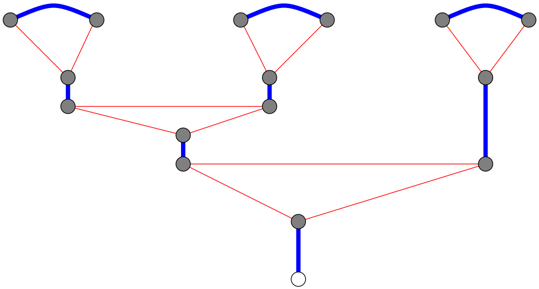


Notice that

- no Æ cycle
- two vertices are always connected via an Æ path

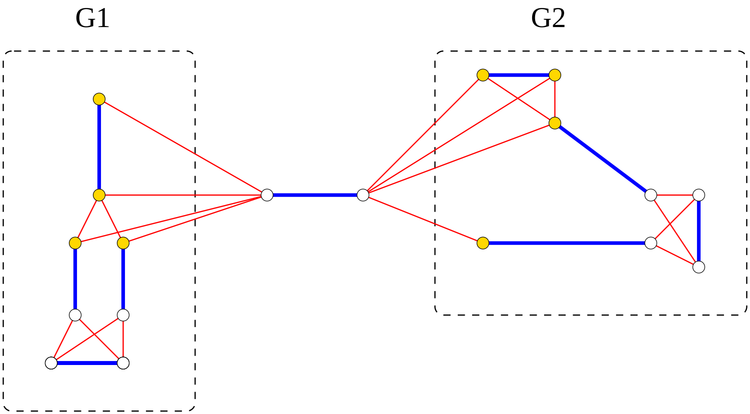
  one does not have MIX : $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ $MIX(A \wedge B) \multimap (A \vee B)$

Example $((a \vee a^{\perp}) \wedge (b \vee b^{\perp})) \wedge (c \vee c^{\perp})$:

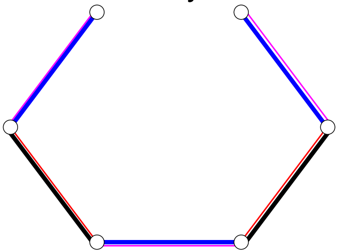


1.2.2. A lemma The following properties are equivalent:

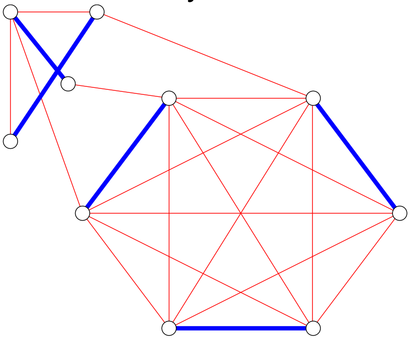
1. no Æ cycle
2. the perfect matching is unique
3. the graph recursively contains a blue bridge

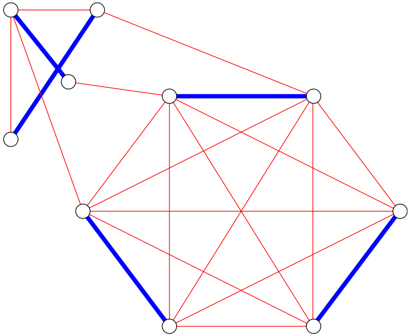


- $\neg 2 \Rightarrow \neg 1$ easy



- $\neg 1 \Rightarrow \neg 2$ easy

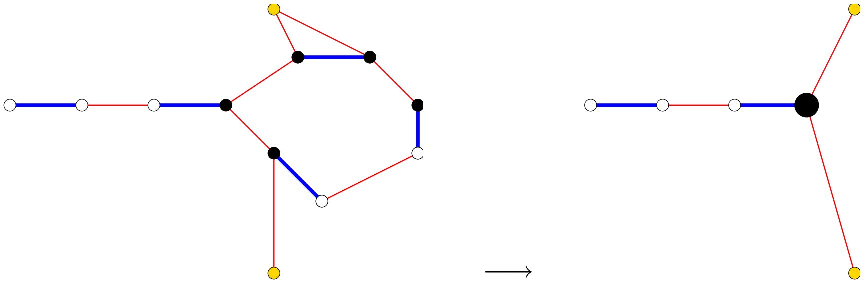




- $3 \Rightarrow 1$ obvious
- $1 \Rightarrow 3$

  extend as much as possible an Æ path

  - pending bridge
  - loop



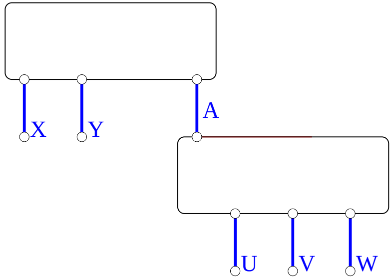
  - Æ path in the reduced graph $\rightarrow$ Æ path in the initial graph
  - bridge in the reduced graph $\rightarrow$ bridge in the initial graph
  - by induction on the size of the graph

1.2.3. Proof net → proof

- One can assume w.l.o.g. that all conclusions of the proofs are conjunctions: (final disjunctions are easy to handle).

- One takes off the final conjunctions, except the red edges between the two premisses
  there exists a blue bridge (lemma)

  - If the bridge is one of the premisses of a conclusion which is a conjunction then the conjunction rule applies



- Otherwise the blue bridge is like this:
  By induction hypothesis we have:
  - a proof δ of $\vdash A^\perp, U, V, W$ one axiom of which is $\vdash A, A^\perp$
  - a proof γ of $\vdash X, Y, A$.

  Replace in δ the axiom $\vdash A, A^\perp$ by the proof of $\vdash X, Y, A$ (replacing $A^\perp$ with X, Y).

$$\dfrac{\dfrac{\overset{\gamma}{\vdash X, Y, A}}{\vdash [A^\perp := X, Y], A} \qquad \vdash \cdots \qquad \vdash \cdots}{\overset{\delta}{\vdash [A^\perp := X, Y], U, V, W}}$$

1.3. Proof nets without links

Idea: identifying more proofs, but not all of them.
Algebraic properties of the connectives:

- associativity
  - $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$
  - $A \vee (B \vee C) \equiv (A \vee B) \vee C$
- commutativity
  - $A \wedge B \equiv B \wedge A$
  - $A \vee B \equiv B \vee A$

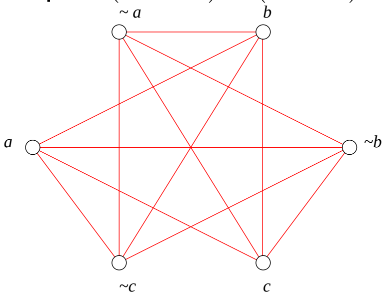
Sequent or disjunction of the formulae in a sequent $\rightarrow$ cograph

- $p \longrightarrow$ vertex (no edges)
  vertices = propositional variables

- $A \vee B \longrightarrow$ disjoint union

- $A \wedge B \longrightarrow$ series composition

Example: $(a \vee a^{\perp}) \wedge (b \vee b^{\perp}) \wedge (c \vee c^{\perp})$



1.3.1. From proofs to graphs

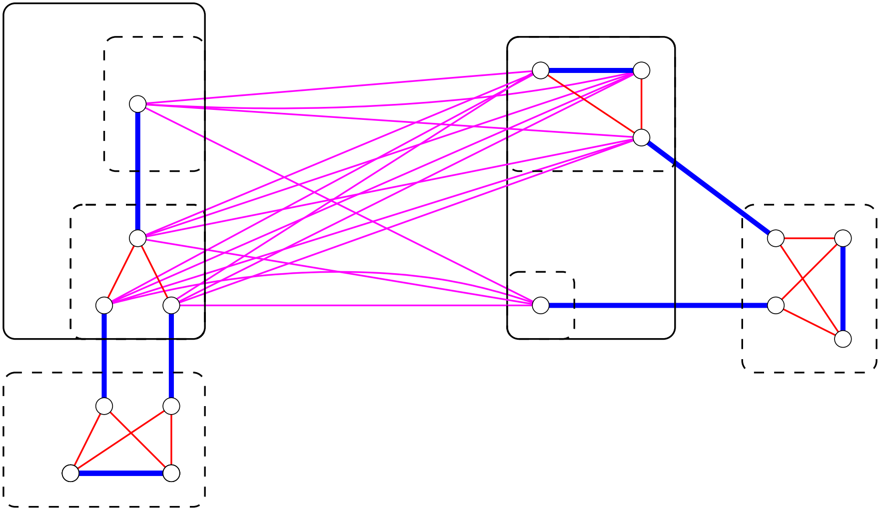
Inductive definition:

- axiom $\longrightarrow$ blue edge

- $\vee$ rule $\longrightarrow$ no effect

- $\wedge$ rule $\rightarrow$ we select a family of connected components in each cograph and we compose them in the series mode

A=p v (a et (b v c)) B=(x et (y v z)) v t



Proof net = cograph + perfect matching

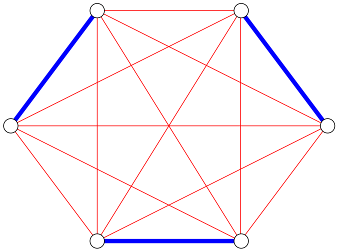
- every Æ cycle contains a chord
- between any two vertices there exists a chordless Æ path (no MIX)

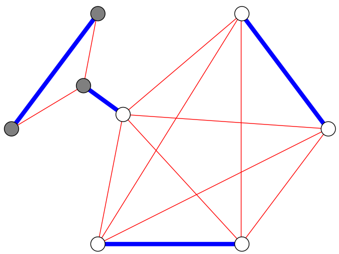
Easy to prove by induction on the number of rules.

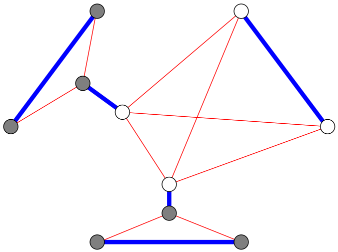
1.3.2. From graphs to proofs One way to prove this is to go back to the previous case, little by little, by considering intermediate structures

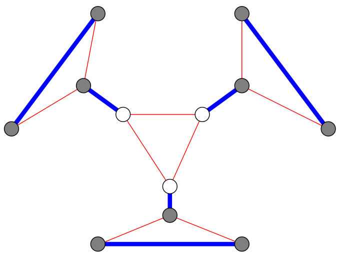
- axioms
- subformulae trees of the conlusions
- cographs whose vertices are conclusions

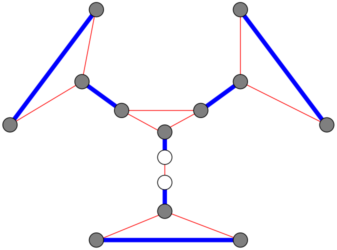
$(a \vee a^{\perp}) \wedge (b \vee b^{\perp}) \wedge (c \vee c^{\perp})$ (correct)

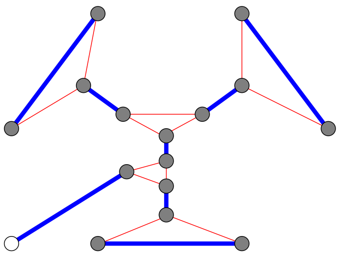












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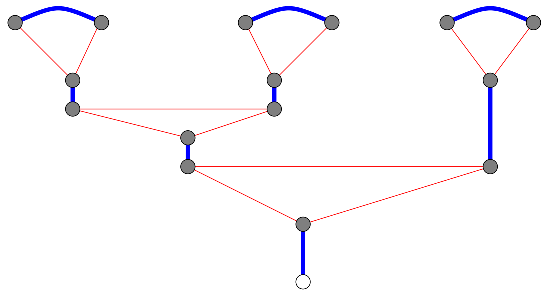


- The transformation and its inverse preserve the criterion:

  - every Æ cycle contains a chord
  - between any two vertices, there exists a chordless Æ path

- Since for a proof net with links there can not be any chord in an Æ path, we have:

- every graph satisfying the criterion corresponds to (at least) one proof.

1.3.3. A definition of proofs by proof net rewriting

Every proof nets is obtained from the complete bicolored graph

$$\wedge_{i\in I}(a_i \vee a_i^\perp) \quad I \text{ multiset } B = \{\ \{a_i, a_i^\perp\} i \in I\}$$

by the following rewrite rule (modulo commutativity and associativity):

$$X \wedge (Y \vee Z) \rightarrow (X \wedge Y) \vee Z$$

If MIX is allowed, add the following rule:

$$X \wedge Y \rightarrow X \vee Y$$

Axiomatisation of cograph inclusion as a rewrite system modulo commutativity and associativity (Bechet, de Groote, Retoré, 97):

$$\begin{array}{rcll} (X \vee X') \wedge (Y \vee Y') & \rightarrow & (X \wedge Y) \vee (X' \wedge Y') & \text{(uncorrect)} \\ X' \wedge (Y \vee Z) & \rightarrow & (X' \wedge Y) \vee Z & \text{(correct)} \\ X' \wedge Y & \rightarrow & X' \vee Y & \text{(correct if MIX is allowed)} \end{array}$$

Funny "coincidence": the rewrite system consisting in the rules which preserve the correctness of the graphs yields all the correct graphs, both for MLL and MLL+MIX.

2. Proof Nets for Cyclic Linear Logic & Lambek Calculus

2.1. Restrictions: non commutativity, intuitionism

Non commutativity No exchange, no crossing: non commutative $\wedge$ and $\vee$. Still some kind of exchange is needed to obtain cut elimination. Cyclic exchange.
Intuitionistism Language restriction: formulae ought to be positive or negative. a positive, $\neg a$ negative. Never perform the conjunction of two negative formulae, nor the disjunction of tow positive formulae. Positive formulae are expressible with implication and conjunction. Negative formulae are the negation of positive fomulae.

In a derivable sequent with only polarisable formulae exactly one is positive, alll the others being negative. Proof of one formula under the negation of the others as hypotheses.
Lambek calculus: non commutative + intuitionistic Provability of some sequents is derivability in a context-free grammar.

2.2. Cyclic Linear Logic

No transposition exchange (no XT)

$$\frac{\vdash \Gamma, A}{\vdash A, \Gamma}\ XC$$

$$\frac{}{\vdash p, p^{\perp}}\ axiom$$

$$\frac{\vdash \Gamma, X \qquad \vdash X^{\perp}, \Gamma'}{\vdash \Gamma, \Gamma'}\ cut$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Gamma'}{\vdash \Gamma, A \wedge B, \Gamma'}\ \wedge$$

$$\frac{\vdash \Gamma, A, B, \Gamma'}{\vdash \Gamma, A \vee B, \Gamma'}\ \vee$$

$$A \wedge B \not\equiv B \wedge A$$

$$A \vee B \not\equiv B \vee A$$

$$(A \wedge B)^{\perp} \equiv B^{\perp} \wedge A^{\perp}$$

2.3. CyMLL Proof Nets with Links (Abrusci & Maringelli)

Red egdes of links are directed.

Criterion:

- the underlying undirected graph is a correct MLL proof net (every Æ path contains a chord)
- there always exist an Æ path from the right premise of a disjunction to its left premise
- there is a black cyclic order between the conclusions, and for every black arc $x \to y$ there is a directed Æ path from $y \longrightarrow x$

2.4. Non commutative proof nets without links

The disjunction has to appear: $(a \wedge b^{\perp}) \vee (b^{\perp} \vee a^{\perp})$ provable but not $(a \wedge b^{\perp}) \vee (a^{\perp} \vee b^{\perp})$.
formulae tree -> two series parallel order

- red: $\wedge$
- black: $\vee$

Sequents of formulae endowed with a cyclic order:

- A black arc from the last atom of a formula to the first of the next formula (in the cyclic order).
- Therefore, a unique red-black Hamiltonian circuit corresponding to the sequence of atoms and cyclic order (yielding ternary relation $H(x, y, z)$, total cyclic order)

Criterion:

- The underlying R&B graph is an MLL proof nets (every æcycle contains a chord, a chordless æcycle between any tow vertices).

- Hamiltonian adequacy B edges define a bracketting wrt. H:

$$\text{If } aBa' \text{ and } bBb' \text{ and } H(a,b,a') \text{ then } H(a,b',a')$$

Result (Pogodalla & Retoré, 2004) Any sequent proof can be turned into a proof net (satisfying the criterion), and conversely alny proof net is the image of at least one sequent proof.

2.5. Partially commutative linear logic

The two kind of connectives, with a relation between them (logical encoding of truly concurrent Petri net execution, grammar with allowed commutativity,...)

Classical system only admit $A \bullet B \rightarrow A \wedge B$. Good proof net syntax (Abrusci & Ruet)

When the calculus is intuitionistic (de Groote, Retoré) is possible to allow as relations bewteen the two kind of connectives the inclusion of series parallel order (rewrite axiomatisation similar to the one for cographs, Bechet, de Groote, Retoré).

Which proof nets? Open question, probably not too difficult.

3. Pomset Logic (Retoré 93)

Another mixed system coming from denotational semantics (coherence spaces a kind of dI-domain): usual associative commutative disjunction and conjunction (MLL) enriched with "before" written "<" associative, non commutative and self-dual ($(A < B)^{\perp} \equiv A^{\perp} < B^{\perp}$, no swap).

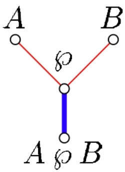
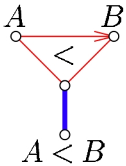
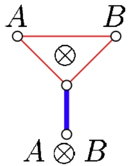
Properties:

- proof nets syntax with or without links (see thereafter)
- cut-elimination (can include a series-parallel order on the cuts, that are the computations to be performed)
- proof nets are exactly the proof structures that can be interpreted within denotational semantics

3.1. Proof nets with links for Pomset logic

Links as in MLL, but an arc $A \rightarrow B$ for $A < B$ possibly an SP order (R arcs as well) bewteen conclusions.

Criterion: no æcircuit (viewing undirected edges as a pair of opposite arcs).

Name	*axiom*-link	*par*-link	*before*-link	*times*-link
Premises	none	A and B	A and B	A and B
R&B-graph				
Conclusions	a and $a^{\perp}$	$A \wp B$	$A < B$	$A \otimes B$

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3.2. Proof nets without links for Pomset logic

Formulae directed cographs:

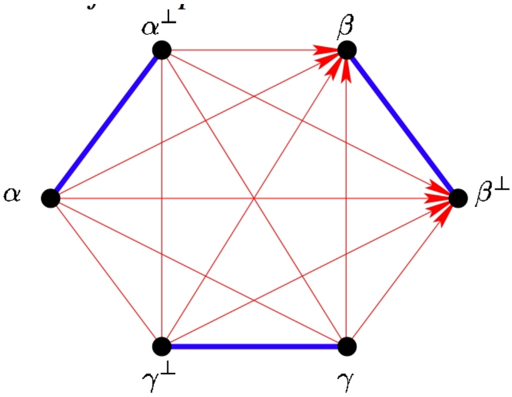
- $\vee$ disjoint union ($\sim$ MLL)
- $\wedge$ series composition (undirected edges) ($\sim$ MLL)
- $<$ series composition (arcs, directed edges)

Characterisation (Bechet, de Groote, Retoré, 97):

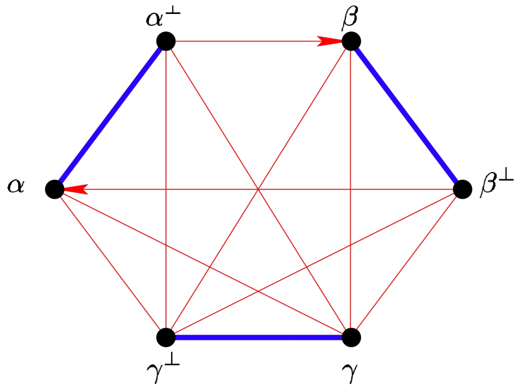
- directed part: SP order,
- undirected part: cograph
- "weak" transitivity: If $(a, b) \in R, (b, c) \in R, (c, b) \in R$ then $(a, c) \in R$ and If $(a, b) \in R, (b, a) \in R, (b, c) \in R$ then $(a, c) \in R$.

Examples of proof structures and nets without links:

Correct

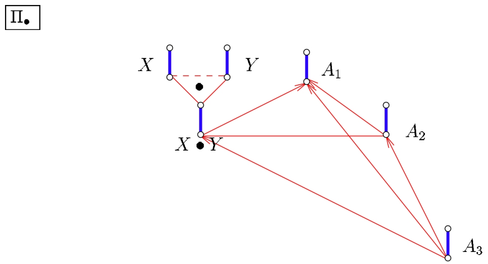


Not correct

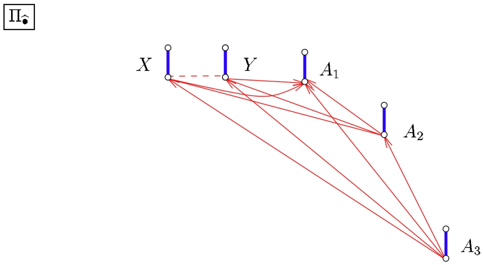


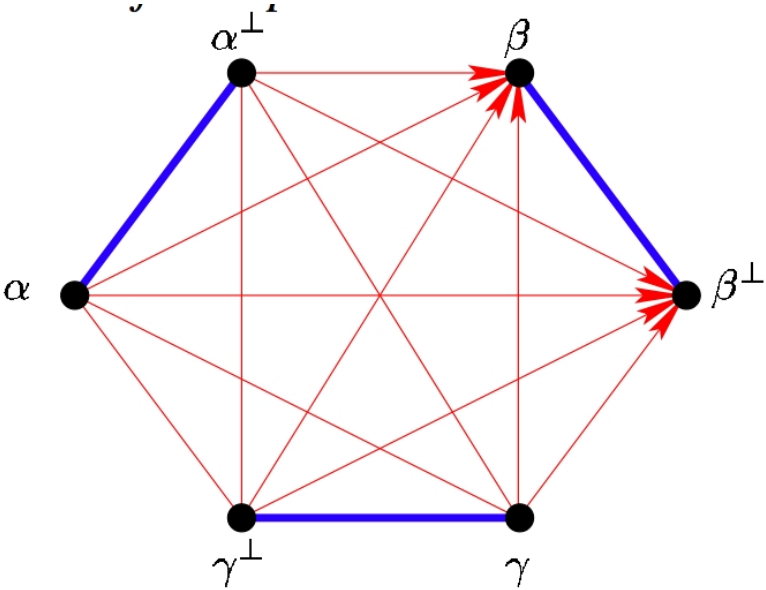
Relation bewteen proof nets with and without links:

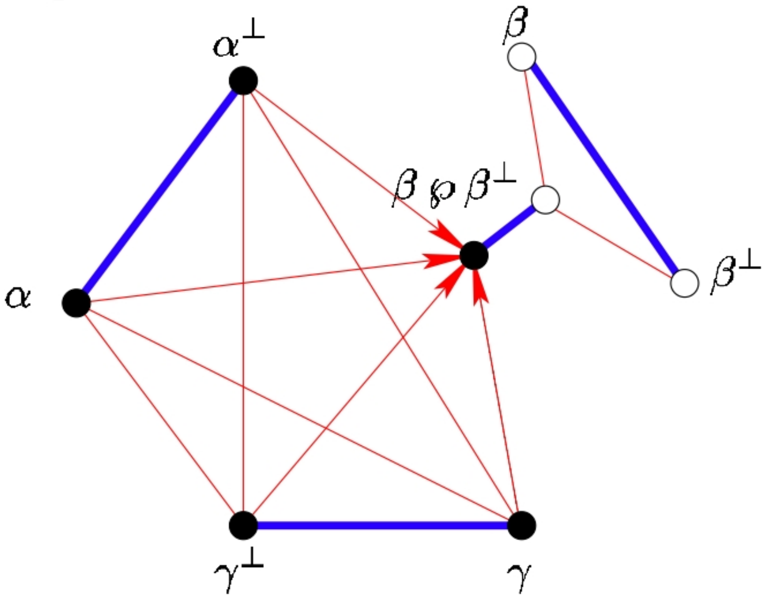
Fold



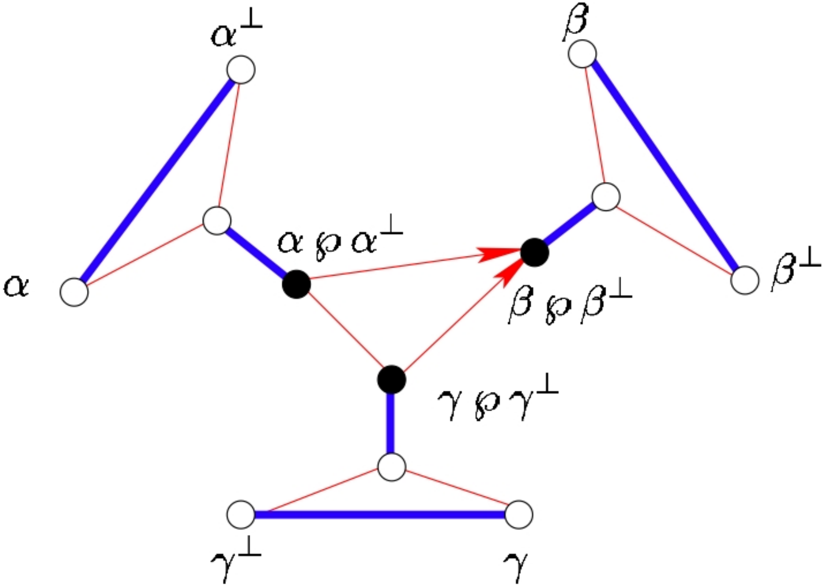
Unfold




$\alpha^\perp$
β
α
$\beta^\perp$
$\gamma^\perp$
γ



$\alpha^{\perp}$
β
$\beta \wp \beta^{\perp}$
α
$\beta^{\perp}$
$\gamma^{\perp}$
γ



$\alpha^\perp$
β
$\alpha \wp \alpha^\perp$
α
$\beta^\perp$
$\beta \wp \beta^\perp$
$\gamma \wp \gamma^\perp$
$\gamma^\perp$
γ

3.3. Sequent calculus proposal, with structured context

Sequents, multisets of formulae, endowed with a series parallel order.

- MIX directed series composition between two proofs.

$$\frac{\vdash \Gamma[I] \qquad \vdash \Delta[J]}{\vdash \Gamma[I < J]}\ MIX$$

- For "relaxing" the order between formulae:

$$\frac{\vdash \Gamma[I]}{\vdash \Gamma[I']}\ ENTROPY(I' \subset I)$$

- TIMES undirected series composition between two unions of R connected components, one in each proof (as usual in MLL) — isolated formulae.
- PAR contraction of equivalent formulae of the SP order.
- BEFORE contraction of lower-equivalent of the SP order.

All the resulting proofs are correct graphs (no chordless Æ-circuit).

Question: are all correct proof nets obtained by these rules?
This can be stated as:

Conjecture 1 Given a graph with no chordless Æ-circuit there exists a partition where the only edges between the two parts are either

- the undirected edges of a single $K_{n,p}$.
- directed edges (arcs) from one part to the other.

3.4. Rewriting axiomatisation

Inclusion of directed cographs is characterised (Bechet, de Groote, Retoré 97) by rewrite rules modulo commutativity and associativity, which are all instances of the interchange rule (a.k.a. Godement law):

$$(X \star X') \bullet (Y \star Y') \rightarrow (X \bullet Y) \star (X \bullet X')$$

These instances are obtained by considering the possibility for a directed cograph to be the empty one (which is the unit wrt. any operation).

Rewrite rules (all correct but the first one)

$(X \vee X') \wedge (Y \vee Y')$	$\rightarrow$	$(X \wedge Y) \vee (X' \wedge Y')$	$(uncorrect)$
$X \wedge (Y \vee Y')$	$\rightarrow$	$(X \wedge Y) \vee Y'$	
$X \wedge Y$	$\rightarrow$	$X \vee Y$	
$(X \vee X') < (Y \vee Y')$	$\rightarrow$	$(X < Y) \vee (X' < Y')$	
$(X \vee X') < Y$	$\rightarrow$	$(X < Y) \vee X'$	
$X < (Y \vee Y')$	$\rightarrow$	$(X < Y) \vee Y'$	
$X' \wedge Y$	$\rightarrow$	$X' \vee Y$	
$(X < X') \wedge (Y < Y')$	$\rightarrow$	$(X \wedge Y) < (X' \wedge Y')$	
$X \wedge (Y < Y')$	$\rightarrow$	$(X \wedge Y) < Y'$	
$(X < X') \wedge Y'$	$\rightarrow$	$X < (X' \wedge Y')$	
$X \wedge Y$	$\rightarrow$	$(X \wedge Y) < (X' \wedge Y')$	

Conjecture 2 Does the correct rewrite rules generate from all the correct proof nets from initial proof nets: $\wedge_{i \in I}(a_i \vee a_i^{\perp}) \quad I$ multiset $B = \{\ \{a_i, a_i^{\perp}\} i \in I\}$?

The two conjectures are equivalent and correspond to the equivalenc ewith the term calculus BV (Strassburger).

4. Conclusion: open questions

- Partially commutative logic We have proof nets without links for cyclic linear logic and for commutative linear logic: can we have similar proof nets for the **intuitionistic** logic involving both, where context are series parallel orders of hypotheses, entropy being the inclusion of series parallel orders? (probably yes)

- Pomset logic Many attempts, a false counter example, a new kind of approach with rewriting rules (as Guglielmi and Strassburger Calculus of Structure, Deep Inference) Would prove the equivalence with their term calculus BV.